

Discrete mean estimates and the Landau-Siegel zero

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Abstract

Let χ be a real primitive character to the modulus D . It is proved that

$$L(1, \chi) \gg (\log D)^{-2022}$$

where the implied constant is absolute and effectively computable.

In the proof, the lower bound for $L(1, \chi)$ is first related to the distribution of zeros of a family of Dirichlet L -functions in a certain region, and some results on the gaps between consecutive zeros are derived. Then, by evaluating certain discrete means of the large sieve type, a contradiction can be obtained if $L(1, \chi)$ is too small.

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1. Introduction

Let χ be a real primitive character to the modulus D . It is known that the Dirichlet L -function $L(s, \chi)$ has at most one real and simple zero $\tilde{\rho}$ satisfying

$$1 - \tilde{\rho} < c_0(\log D)^{-1}$$

where $c_0 > 0$ is an absolute constant. Such a zero is called the Landau-Siegel zero. The typical methods to determine zero-free regions for Dirichlet L -functions are unable to eliminate the Landau-Siegel zero for an intrinsic reason.

The non-vanishing of $L(s, \chi)$ near $s = 1$ is closely related to the lower bound for the value of $L(s, \chi)$ at $s = 1$. The well-known Siegel theorem [19] asserts that, for any $\varepsilon > 0$, there exists a positive number $C_1(\varepsilon)$ such that

$$L(1, \chi) > C_1(\varepsilon)D^{-\varepsilon}.$$

This implies, for any $\varepsilon > 0$, that there exists a positive number $C_2(\varepsilon)$ such that

$$L(\sigma, \chi) \neq 0 \quad \text{if} \quad \sigma > 1 - C_2(\varepsilon)D^{-\varepsilon}.$$

However, the process of Siegel's proof essentially involves certain assumptions, such as the existence of a real primitive character $\hat{\chi} \pmod{\hat{D}}$ with $L(s, \hat{\chi})$ vanishing on the interval $[1 - \varepsilon, 1)$ (see [8]), and the constant $C_1(\varepsilon)$ relies on the unknown modulus \hat{D} necessarily. This makes the result of Siegel's theorem entirely ineffective.

It is known (see [9]) that the non-existence of the Landau-Siegel zero implies

$$L(1, \chi) \gg (\log D)^{-1}. \tag{1.1}$$

In the case $\chi(-1) = -1$, Goldfeld [10] and Gross and Zagier [12] proved that

$$L(1, \chi) \gg D^{-1/2}(\log D)^{1-\varepsilon}$$

for any $\varepsilon > 0$, where the implied constant is effectively computable. This result surpasses the trivial bound $L(1, \chi) \gg D^{-1/2}$ and solves Gauss' class number problem for imaginary quadratic fields. Granville and Stark [11] proved that the uniform *abc* conjecture for number fields implies (1.1) when $\chi(-1) = -1$.

The main result of this paper is

Theorem 1 *If χ is a real primitive character to the modulus D , then*

$$L(1, \chi) > c_1(\log D)^{-2022}$$

where $c_1 > 0$ is an absolute, effectively computable constant.

As a direct consequence of Theorem 1 we have

Theorem 2 *If χ is a real primitive character to the modulus D , then*

$$L(\sigma, \chi) \neq 0$$

for

$$\sigma > 1 - c_2(\log D)^{-2024}$$

where $c_2 > 0$ is an absolute, effectively computable constant.

It is possible to replace the exponent -2022 in Theorem 1 by a larger (negative) value if the current arguments are modified, but we will not discuss it in this paper. On the other hand, it seems that the lower bound (1.1) can not be achieved by the present methods.

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2. Notation and outline of the proof

Notation and conventions.

Throughout, t, u, v, z and σ denote real variables; x and y denote positive real variables; s and w denote the complex variables $s = \sigma + it$ and $w = u + iv$; d, h, j, k, l, m, n and r denote natural numbers, while p and q denote primes; the arithmetic functions

$\mu(n)$ and $\varphi(n)$ are defined as usual, while $\tau_k(n)$ denotes the k -fold divisor function. Let $*$ denote the Dirichlet convolution of arithmetic functions. Let $e(s) = \exp\{2\pi is\}$. We write

$$\sum_n \quad \text{and} \quad \sum_{(n,k)=1}$$

for

$$\sum_{n=1}^{\infty} \quad \text{and} \quad \sum_{\substack{n=1 \\ (n,k)=1}}^{\infty}$$

respectively. Let $\int_{(z)}$ denote an integration from $z - i\infty$ to $z + i\infty$.

We let χ denote a real primitive character to the modulus D with D greater than a sufficiently large and effectively computable number. Write

$$\mathcal{L} = \log D. \tag{2.1}$$

Let c denote a positive, effectively computable and absolute constant, not necessarily the same at each occurrence. All implied constants, unless specified, are absolute and effectively computable. In this paper, an error term is usually given in the form that is small enough for our purpose, although sharper estimates are possible.

For any Dirichlet character θ to the modulus k , let $\tau(\theta)$ denote the Gauss sum

$$\tau(\theta) = \sum_{a \pmod{k}} \theta(a)e(a/k)$$

(thus we shall not write τ for τ_2 , the ordinary divisor function). The expressions

$$\sum_{\theta \pmod{k}}, \quad \sum'_{\theta \pmod{k}} \quad \text{and} \quad \sum^*_{\theta \pmod{k}}$$

denote, respectively, a sum over all $\theta \pmod{k}$, a sum over all non-principal $\theta \pmod{k}$, and a sum over all primitive $\theta \pmod{k}$. Let ψ_k^0 denote the principle character \pmod{k} .

In case $\theta \pmod{k}$ is a primitive character, the functional equation for $L(s, \theta)$ is

$$L(s, \theta) = Z(s, \theta)L(1 - s, \bar{\theta}), \tag{2.2}$$

where

$$Z(s, \theta) = \tau(\theta)\pi^{s-1/2}k^{-s} \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)^{-1}$$

if $\theta(-1) = 1$, and

$$Z(s, \theta) = -i\tau(\theta)\pi^{s-1/2}k^{-s} \Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)^{-1}$$

if $\theta(-1) = -1$. When t is large, it is convenient to use an asymptotic expression for $Z(s, \theta)$ as follows. Let

$$\vartheta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s/2) \quad (2.3)$$

(this function is usually written as $\chi(s)$ in the literature). It is known that

$$\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)^{-1} = \vartheta(s)$$

and

$$\pi^{s-1/2}\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)^{-1} = \vartheta(s)\cot(\pi s/2).$$

Assume $t > 1$. Then

$$i\cot(\pi s/2) = 1 + O(e^{-\pi t}).$$

Thus we have uniformly

$$Z(s, \theta) = \theta(-1)\tau(\theta)k^{-s}\vartheta(s)(1 + O(e^{-\pi t})), \quad (2.4)$$

and

$$Z(s, \theta)^{-1} = \tau(\bar{\theta})k^{s-1}\vartheta(1-s)(1 + O(e^{-\pi t})), \quad (2.5)$$

the $O(e^{-\pi t})$ term being identically zero in case $\theta(-1) = 1$.

Outline of the proof: Initial steps

Our aim is to derive a contradiction from the following

Assumption (A)

$$L(1, \chi) < \mathcal{L}^{-2022}.$$

The underlying ideas are inspired by the work of Goldfeld [10], Iwaniec and Sarnak [15] and Conrey, Ghosh and Gonek [4]. The paper [10] exhibits a relationship between the lower bound for $L(1, \chi)$ and the order of zeros of the function $L_E(s)L_E(s, \chi)$ at the central point, where E is an elliptic curve; the paper [15] exhibits a relationship between the lower bound for $L(1, \chi)$ and the non-vanishing of central values of a family of automorphic L -functions; the paper [4] has definite influence on the idea of reducing the problem to evaluating certain discrete means that is employed in the present work (the reader is also referred to [3]).

The initial part of this paper consists in relating the lower bound for $L(1, \chi)$ to the distribution of zeros of the function $L(s, \psi)L(s, \psi\chi)$, with ψ belonging to a large set Ψ of primitive characters, in a domain Ω . Thus we introduce

$$P = \exp\{\mathcal{L}^9\}. \quad (2.6)$$

For notational simplicity we write

$$p \sim P \quad \text{for} \quad P < p < P(1 + \mathcal{L}^{-68}).$$

Let Ψ be the set of all primitive characters $\psi(\bmod p)$ with $p \sim P$, and let

$$\Omega = \{s : |\Re(s - s_0)| < 1/2, \quad |\Im(s - s_0)| < \mathcal{L}_1 + 2\} \quad (2.7)$$

where

$$\mathcal{L}_1 = \mathcal{L}^{405}, \quad s_0 = \frac{1}{2} + 2\pi it_0 \quad \text{with} \quad t_0 = \mathcal{L}^{519}. \quad (2.8)$$

Throughout, we will assume that $p \sim P$ and ψ is a primitive characters $(\bmod p)$, i.e., $\psi \in \Psi$. Note that

$$\mathcal{P} := \sum_{p \sim P} p = (1 + o(1))P^2 \mathcal{L}^{-77}. \quad (2.9)$$

In Section 3 we shall introduce a subset Ψ_1 of Ψ and prove

Proposition 2.1. *Let Ψ_2 be the complement of Ψ_1 in Ψ . If (A) holds, then*

$$\sum_{\psi \in \Psi_2} 1 \ll \mathcal{P} \mathcal{L}^{-739}.$$

For $\psi(\bmod p) \in \Psi$, the average gap between consecutive zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω is

$$\simeq \frac{2\pi}{\log(Dp^2 t_0^2)} = \alpha(1 + O(\alpha \mathcal{L})).$$

where

$$\alpha = \frac{\pi}{\log P}. \quad (2.10)$$

In Section 4 we shall prove

Proposition 2.2. *Suppose that $\psi \in \Psi_1$. Then the following hold.*

- (i). *All the zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω lie on the critical line.*
- (ii). *All the zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω are simple.*
- (iii). *The gap between any consecutive zeros of $L(s, \psi)L(s, \psi\chi)$ in Ω is of the form*

$$\alpha + O(\alpha^2 \mathcal{L}).$$

It should be stressed that the set Ψ_1 is formally defined. In fact, Proposition 2.2 and some other results for $\psi \in \Psi_1$ are unconditionally derived from the definition of Ψ_1 .

However, one is even unable to determine whether Ψ_1 is empty or not without assuming (A).

The results of Proposition 2.1 and 2.2 are not surprising and they go back to Heath-Brown [15]. One may believe that the assertions (i) and (ii) of Proposition 2.2 hold for all characters ψ in Ψ . On the other hand, however, in comparison with some conjectures on the vertical distribution of zeros of the Riemann zeta function (see [2] and [17]), one may claim that the gap assertion (iii) of Proposition 2.2 fails to hold for most of ψ in Ψ . Our proof thus consists in deriving a contradiction from the gap assertion.

Assume $\psi \in \Psi$. Recall that the functional equation for $L(s, \psi)$ is given by (2.2) with $\theta = \psi$. On the upper-half plane, since $Z(s, \psi)$ is analytic and non-vanishing, there is an analytic function $Y(s, \psi)$ such that

$$Y(s, \psi)^2 = Z(s, \psi)^{-1}.$$

Let

$$M(s, \psi) = Y(s, \psi)L(s, \psi).$$

Note that $M(s, \psi)$ is defined for $t > 0$ only, and, for each ψ , there are two choices of $M(s, \psi)$ up to \pm , but the expressions

$$M(s_1, \psi)M(s_2, \psi) \quad \text{and} \quad \frac{M(s_1, \psi)}{M(s_2, \psi)}$$

are independent of these choices. The functional equation (2.2) gives

$$M(s, \psi) = Y(s, \psi)^{-1}L(1 - s, \bar{\psi}).$$

Since $|Y(1/2 + it, \psi)| = 1$, it follows that

$$|M(1/2 + it, \psi)| = |L(1/2 + it, \psi)|,$$

$$M(1/2 + it, \psi) \in \mathbf{R}, \tag{2.11}$$

and, consequently,

$$iM'(1/2 + it, \psi) \in \mathbf{R}. \tag{2.12}$$

The gap assertion (iii) of Proposition 2.2 can be restated as follows. If $\psi \in \Psi_1$ and $(1/2 + i\gamma, 1/2 + i\gamma')$ is a pair of consecutive zeros of $L(s, \psi)L(s, \chi\psi)$ in Ω with $\gamma' > \gamma$, then for some (large) constant $c' > 0$,

$$|\gamma' - \gamma - \alpha| < c'\alpha^2\mathcal{L}.$$

Write

$$\beta_1 = i\alpha(1 - 5c'\alpha\mathcal{L}), \quad \beta_2 = 2i\alpha(1 + c'\alpha\mathcal{L}), \quad \beta_3 = 3i\alpha(1 - c'\alpha\mathcal{L}). \tag{2.13}$$

In what follows we assume $\psi \in \Psi_1$. Let $\mathcal{Z}(\psi)$ denote the set of zeros of $L(s, \psi)$ in the region

$$|\sigma - 1/2| < 1/2, \quad |t - 2\pi t_0| < \mathcal{L}_1 \quad (2.14)$$

(this is slightly smaller than Ω). Assume $\rho \in \mathcal{Z}(\psi)$. Note that

$$M'(\rho, \psi) = Y(\rho, \psi)L'(\rho, \psi) \neq 0.$$

Write

$$\mathcal{C}^*(\rho, \psi) = \frac{-iM(\rho + \beta_1, \psi)M(\rho + \beta_2, \psi)M(\rho + \beta_3, \psi)}{M'(\rho, \psi)}.$$

By (2.11) and (2.12) we have

$$\mathcal{C}^*(\rho, \psi) \in \mathbf{R}.$$

It will be proved that

Lemma 2.3. *Suppose $\psi \in \Psi_1$ and $\rho \in \mathcal{Z}(\psi)$. Then*

$$\mathcal{C}^*(\rho, \psi) \geq 0.$$

We introduce the smooth weight

$$\omega(s) = \frac{\sqrt{\pi}}{\mathcal{L}_2} \exp \left\{ \frac{(s - s_0)^2}{4\mathcal{L}_2^2} \right\} \quad \text{with} \quad \mathcal{L}_2 = \mathcal{L}^{400}, \quad (2.15)$$

which is positive for $\sigma = 1/2$. Lemma 2.3 implies that

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |\mathcal{H}(\rho, \psi)|^2 \omega(\rho) \geq 0. \quad (2.16)$$

for all functions $\mathcal{H}(s, \psi)$ defined on Ω . Thus, in order to derive a contradiction from (A), it suffices to find certain functions $\mathcal{H}(s, \psi)$ and show that, under Assumption (A), (2.16) fails to hold.

A heuristic argument

It is natural to consider the function $\mathcal{H}(s, \psi)$ which is of the form

$$\sum_{n < P'} \frac{f(\log n / \log P') \mu \psi(n)}{n^s}$$

for some $P' < P$, where f is a polynomial with $f(1) = 0$. It should be stressed, on assuming (A), that if $\mathcal{H}(s, \psi)$ is of the above form with P' slightly smaller than P in the logarithmic scale, the sum in (2.16) can be evaluated. This is analogous to the results of Conrey, Iwaniec and Soundararajan [5] and [6]. However, it seems that such a choice of

$\mathcal{H}(s, \psi)$ is not good enough for our purpose. On the other hand, the left side of (2.16) could be “close” to zero for certain non-zero functions $\mathcal{H}(s, \psi)$. For example, one may consider the choice that $\mathcal{H}(s, \psi)$ is a linear combination of $L(s + \beta_j, \chi\psi)$, $1 \leq j \leq 3$. Thus, inspired by the approximate formula for $L(s, \chi\psi)$, some new forms of $\mathcal{H}(s, \psi)$ could be introduced. However, our goal is not to show that (2.16) fails to hold for certain $\mathcal{H}(s, \psi)$ directly, instead, a variant of the argument will be adopted.

The crucial part of the proof is to construct functions $H_1(s, \psi)$, $H_2(s, \psi)$, $J_1(s, \psi)$ and $J_2(s, \psi)$, each of which is of the form

$$\sum_{n < P'} \frac{f(\log n / \log P) \chi\psi(n)}{n^s}$$

with $\log P' / \log P$ close to $1/2$, and show that the left side of (2.16) is “small” if

$$\mathcal{H}(s, \psi) = H_1(s, \psi) + Z(s, \chi\psi) \overline{H_2(s, \psi)} \quad \text{or} \quad \mathcal{H}(s, \psi) = J_1(s, \psi) + Z(s, \chi\psi) \overline{J_2(s, \psi)}$$

for $\sigma = 1/2$. On the other hand, the sum

$$\Xi_1^* := \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) (H_1(\rho, \psi) \overline{J_1(\rho, \psi)} + \overline{H_2(\rho, \psi)} J_2(\rho, \psi)) \omega(\rho) \quad (2.17)$$

is not too small in absolute value.

For $\psi \in \Psi_1$ and $\rho \in \tilde{\mathcal{Z}}(\psi)$, since

$$\begin{aligned} & H_1(\rho, \psi) \overline{J_1(\rho, \psi)} + \overline{H_2(\rho, \psi)} J_2(\rho, \psi) \\ &= (H_1(\rho, \psi) + Z(\rho, \chi\psi) \overline{H_2(\rho, \psi)}) \overline{J_1(\rho, \psi)} - (Z(\rho, \chi\psi) \overline{J_1(\rho, \psi)} - J_2(\rho, \psi)) \overline{H_2(\rho, \psi)} \end{aligned}$$

and

$$|Z(\rho, \chi\psi) \overline{J_1(\rho, \psi)} - J_2(\rho, \psi)| = |J_1(\rho, \psi) - Z(\rho, \chi\psi) \overline{J_2(\rho, \psi)}|,$$

it follows that

$$\begin{aligned} & |H_1(\rho, \psi) \overline{J_1(\rho, \psi)} + \overline{H_2(\rho, \psi)} J_2(\rho, \psi)| \\ & \leq |H_1(\rho, \psi) + Z(\rho, \chi\psi) \overline{H_2(\rho, \psi)}| |J_1(\rho, \psi)| + |J_1(\rho, \psi) - Z(\rho, \chi\psi) \overline{J_2(\rho, \psi)}| |H_2(\rho, \psi)|. \end{aligned}$$

This yields, by Lemma 2.3,

$$|\Xi_1^*| \leq \Xi_2^* + \Xi_3^* \quad (2.18)$$

where

$$\Xi_2^* = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |H_1(\rho, \psi) + Z(\rho, \chi\psi) \overline{H_2(\rho, \psi)}| |J_1(\rho, \psi)| \omega(\rho), \quad (2.19)$$

$$\Xi_3^* = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |J_1(\rho, \psi) - Z(\rho, \chi\psi) \overline{J_2(\rho, \psi)}| |H_2(\rho, \psi)| \omega(\rho). \quad (2.20)$$

Discrete mean estimates

We introduce some parameters and functions as follows. Let

$$P_1 = P^{0.504}, \quad P_2 = P^{0.5}T^{-10}, \quad P_3 = P^{0.498}, \quad (2.21)$$

$$\beta_6 = \frac{3i\alpha}{2}, \quad \beta_7 = \frac{5i\alpha}{2}. \quad (2.22)$$

(The definition of β_4 and β_5 will be given in Section 8.) Write

$$H_{11}(s, \psi) = \sum_{n < P_1} \frac{\chi\psi(n)}{n^s} \left(1 - \frac{\log n}{\log P_1}\right) \left(\frac{P_1}{n}\right)^{\beta_6}, \quad (2.23)$$

$$H_{12}(s, \psi) = \sum_{n < P_2} \frac{\chi\psi(n)}{n^s} \left(1 - \frac{\log n}{\log P_2}\right) \left(\frac{P_2}{n}\right)^{\beta_7}, \quad (2.24)$$

$$H_{13}(s, \psi) = \sum_{n < P_3} \frac{\chi\psi(n)}{n^s} \left(1 - \frac{\log n}{\log P_3}\right) \left(\frac{P_3}{n}\right)^{\beta_6} \quad (2.25)$$

(these sums are written as $H_{1j}(s, \psi)$ rather than $H_{1j}(s, \chi\psi)$). We also introduce the numerical constants

$$\iota_2 = 0.94977 - 1.38995i, \quad \iota_3 = -1.00635 - 0.22789i, \quad \iota_4 = -0.68738 + 1.60688i. \quad (2.26)$$

We define

$$H_1(s, \psi) = H_{11}(s, \psi) + \iota_2 H_{12}(s, \psi), \quad H_2(s, \psi) = \bar{\iota}_3 H_{13}(s, \psi) + \bar{\iota}_4 H_{12}(s, \psi). \quad (2.27)$$

Let $\tilde{f}(z)$ be given by

$$\tilde{f}(z) = \begin{cases} 500(z - 0.5) & \text{if } 0.5 \leq z \leq 0.502 \\ 500(0.504 - z) & \text{if } 0.502 \leq z \leq 0.504 \\ 0 & \text{otherwise.} \end{cases} \quad (2.28)$$

We define

$$J_1(s, \psi) = \sum_n \frac{\chi\psi(n)}{n^s} \tilde{f}\left(\frac{\log n}{\log P}\right), \quad (2.29)$$

$$J_2(s, \psi) = \sum_n \frac{\chi\psi(n)}{n^s} \tilde{f}\left(\frac{\log n}{\log P} + 0.004 - \tilde{\alpha}\right) \quad \text{with} \quad \tilde{\alpha} = \frac{\log Dt_0}{\log P}. \quad (2.30)$$

Let

$$\mathbf{a} = \frac{6}{\pi^2} L'(1, \chi)^2 \prod_{q|D} \frac{q}{q+1}. \quad (2.31)$$

It can be shown that the assumption (A) implies

$$\mathfrak{a} \gg 1.$$

(see Lemma 5.7).

Proposition 2.4. *Assume that (A) holds. Then*

$$|\Xi_1^*| > 5\mathfrak{a}\mathcal{P}.$$

Proposition 2.5. *Assume that (A) holds. Then*

$$\Xi_2^* < 2\mathfrak{a}\mathcal{P}.$$

Proposition 2.6. *Assume that (A) holds. Then*

$$\Xi_3^* = o(\mathfrak{a}\mathcal{P}).$$

Under Assumption (A), a contradiction is immediately derived from (2.18), Proposition 2.4, 2.5 and 2.6, This proves Theorem 1.

It should be remarked that Proposition 2.5 follows from the inequalities (on assuming (A))

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |H_1(\rho, \psi) + Z(\rho, \chi\psi) \overline{H_2(\rho, \psi)}|^2 \omega(\rho) < 0.001\mathfrak{a}\mathcal{P}, \quad (2.32)$$

and

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |J_1(\rho, \psi)|^2 \omega(\rho) < 3000\mathfrak{a}\mathcal{P}, \quad (2.33)$$

by Cauchy's inequality (and Lemma 2.3). The proof of (2.32) involves some numerical calculations.

We conclude this section by proving Lemma 2.3.

Proof of Lemma 2.3. By (2.) we see that $M(\rho + iv, \psi) \neq 0$ if $|\beta_2| \leq v \leq |\beta_3|$. This implies, by the mean-value theorem, that

$$M(\rho + \beta_2, \psi)M(\rho + \beta_3, \psi) > 0,$$

since $M(1/2 + it, \psi)$ is a real-valued continuous function in t . Similarly we have

$$\frac{M(\rho + \beta_1, \psi)}{M(\rho + iv, \psi)} > 0$$

if $0 < v \leq |\beta_1|$. Since

$$\frac{M(\rho + \beta_1, \psi)}{iM'(\rho, \psi)} = \lim_{v \rightarrow 0^+} \frac{vM(\rho + \beta_1, \psi)}{M(\rho + iv, \psi)},$$

it follows that

$$\frac{M(\rho + \beta_1, \psi)}{iM'(\rho, \psi)} \geq 0.$$

This completes the proof. \square

Remark. It is implied in the proof of Lemma 2.3 that

$$\left| \frac{L(\rho + \beta_1, \psi)}{L'(\rho, \psi)} \right| = -i \frac{M(\rho + \beta_1, \psi)}{M'(\rho, \psi)} \quad (2.34)$$

for $\psi \in \Psi_1$ and $\rho \in \tilde{Z}(\psi)$.

3. The set Ψ_1

Let $\nu(n)$ and $v(n)$ be given by

$$\zeta(s)L(s, \chi) = \sum_n \frac{\nu(n)}{n^s}, \quad \zeta(s)^{-1}L(s, \chi)^{-1} = \sum_n \frac{v(n)}{n^s}, \quad \sigma > 1,$$

respectively. It is easy to see that

$$|v(n)| \leq \nu(n) \leq \tau_2(n). \quad (3.1)$$

Lemma 3.1. *Assume (A) holds. Then*

$$\sum_{D^4 < n \leq P^2} \frac{\nu(n)^2}{n} \ll \mathcal{L}^{-2011}. \quad (3.2)$$

Proof. Let

$$\phi(s) = \zeta(s)^{-2}L(s, \chi)^{-2} \sum_n \frac{\nu(n)^2}{n^s}$$

which has the Euler product representation

$$\phi(s) = \prod_p \phi_p(s) \quad (\sigma > 1).$$

For $\sigma \geq \sigma_0 > 0$, by checking the cases $\chi(p) = \pm 1$ and $\chi(p) = 0$ respectively, it can be seen that

$$\phi_p(s) = 1 + O(p^{-2\sigma}) \quad \text{if } p \nmid D,$$

and

$$\phi_p(s) = (1 - p^{-s})(1 + O(p^{-2\sigma})) \quad \text{if } p|D,$$

the implied constant depending on σ_0 . Thus $\phi(s)$ is analytic for $\sigma > 1/2$ and it satisfies

$$\phi(s) \ll \prod_{p|D} |1 - p^{-s}| \quad (3.3)$$

for $\sigma \geq \sigma_1 > 1/2$, the implied constant depending on σ_1 . The left side of (3.2) is

$$\begin{aligned} &\ll \sum_n \frac{\nu(n)^2}{n} (\exp\{-n/P^2\} - \exp\{-n/D^4\}) \\ &= \frac{1}{2\pi i} \int_{(1)} \phi(1+s) \zeta(1+s)^2 L(1+s, \chi)^2 (P^{2s} - D^{4s}) \Gamma(s) ds. \end{aligned}$$

Moving the line of integration to the left appropriately and applying standard estimates, we see that the right side is equal to the residue of the integrand plus an acceptable error $O(D^{-c})$. The residue at $s = 0$ can be written as

$$\frac{1}{2\pi i} \int_{|s|=\alpha^*} \phi(1+s) \zeta(1+s)^2 L(1+s, \chi)^2 (P^{2s} - D^{4s}) \Gamma(s) ds$$

with $\alpha^* = \mathcal{L}^{-2024}$. If $|s| = \alpha^*$, then $\phi(1+s) = O(1)$ by (3.3),

$$L(1+s, \chi) \ll \mathcal{L}^{-2022}$$

by (A) and standard estimates, and

$$\zeta(1+s) \ll \mathcal{L}^{2024}, \quad (P^{2s} - D^{4s}) \Gamma(s) \ll \mathcal{L}^9.$$

It follows that the residue at $s = 0$ is $\ll \mathcal{L}^{-2011}$, whence the result follows. \square

Lemma 3.2. *Assume (A) holds. Then we have*

$$\sum_{D^4 < n \leq D^8} \frac{\nu(n)^2 \tau_2(n)^2}{n} \ll \mathcal{L}^{-2007}.$$

Proof. As the situation is analogous to Lemma 3.1 we give a sketch only. It can be verified that the function

$$\phi^*(s) = \zeta(s)^{-8} L(s, \chi)^{-8} \sum_n \frac{\nu(n)^2 \tau_2(n)^2}{n^s}$$

is analytic for $\sigma > 1/2$ and it satisfies

$$\phi^*(s) \ll \prod_{p|D} |1 - p^{-4s}|$$

for $\sigma \geq \sigma_1 > 1/2$, the implies constant depending on σ_1 . Also, one can verify that

$$\int_{|s|=\alpha^*} \phi^*(1+s) \zeta(1+s)^8 L(1+s, \chi)^8 (D^{8s} - D^{4s}) \Gamma(s) ds \ll \mathcal{L}^{-2007}.$$

This completes the proof. \square

Lemma 3.3. *For any s and any complex numbers $c(n)$ we have*

$$\sum_{\psi \in \Psi} \left| \sum_{n \leq P} \frac{c(n)\psi(n)}{n^s} \right|^2 \ll \mathcal{P} \sum_{n \leq P} \frac{|c(n)|^2}{n^{2\sigma}}$$

and

$$\sum_{\psi \in \Psi} \left| \sum_{n \leq P^2} \frac{c(n)\psi(n)}{n^s} \right|^2 \ll P^2 \sum_{n \leq P^2} \frac{|c(n)|^2}{n^{2\sigma}}.$$

Proof. The first assertion follows by the orthogonality relation; the second assertion follows by the large sieve inequality. \square

Let

$$F(s, \psi) = \sum_{n \leq D^4} \frac{\nu(n)\psi(n)}{n^s}, \quad G(s, \psi) = \sum_{n \leq D^4} \frac{v(n)\psi(n)}{n^s}.$$

By (3.1) we may write

$$F(s, \psi)^{20} = \sum_{n \leq D^{80}} \frac{\nu_{20}(n)\psi(n)}{n^s}, \quad G(s, \psi)^{20} = \sum_{n \leq D^{80}} \frac{v_{20}(n)\psi(n)}{n^s}$$

with $|\nu_{20}n| \leq \tau_{40}(n)$, $|v_{20}n| \leq \tau_{40}(n)$. Write

$$X_1(x, \psi) = \sum_{n \leq x} \frac{\nu_{20}(n)\psi(n)}{n^{s_0}}, \quad X_2(x, \psi) = \sum_{n \leq x} \frac{v_{20}(n)\psi(n)}{n^{s_0}}.$$

By Cauchy's inequality and the first assertion of Lemma 3.3 we obtain

$$\sum_{\psi \in \Psi} \left(|X_1(D^{80}, \psi)| + |X_2(D^{80}, \psi)| + \int_1^{D^{80}} \frac{|X_1(x, \psi)| + |X_2(x, \psi)|}{x} dx \right)^2 \ll \mathcal{P} \mathcal{L}^{1602}.$$

Thus we conclude

Lemma 3.4. *The inequality*

$$|X_1(D^{80}, \psi)| + |X_2(D^{80}, \psi)| + \int_1^{D^{80}} \frac{|X_1(x, \psi)| + |X_2(x, \psi)|}{x} dx < \mathcal{L}^{1171} \quad (3.4)$$

holds for all but at most $O(\mathcal{P}\mathcal{L}^{-740})$ characters ψ in Ψ .

Write

$$X_3(x, \psi) = \sum_{D^4 < n \leq x} \frac{\nu(n)\psi(n)}{n^{80}} \quad \text{for } x > D^4.$$

Assume that (A) holds. By Cauchy's inequality, the second assertion of Lemma 3.2 and Lemma 3.1,

$$\sum_{\psi \in \Psi} \left(|X_3(P^2, \psi)| + \int_{D^4}^{P^2} \frac{|X_3(x, \psi)|}{x} dx \right)^2 \ll P^2 \mathcal{L}^{-1993} \ll \mathcal{P}\mathcal{L}^{-1909}.$$

Thus we conclude

Lemma 3.5 *Assume that (A) holds. The inequality*

$$|X_3(P^2, \psi)| + \int_{D^4}^{P^2} \frac{|X_3(x, \psi)|}{x} dx < \mathcal{L}^{-585} \quad (3.5)$$

holds for all but at most $O(\mathcal{P}\mathcal{L}^{-739})$ characters ψ in Ψ .

Let

$$\varsigma(n) = \sum_{\substack{n=lm \\ l, m \leq D^4}} \nu(l)\nu(m).$$

We have $\varsigma(n) = 0$ unless $n = 1$ or $D^4 < n \leq D^8$. Write

$$X_4(x, \psi) = \sum_{D^4 < n \leq x} \frac{\varsigma(n)\psi(n)}{n^{80}}$$

for $x > D^4$. It is direct to verify that

$$|\varsigma(n)| \leq \sum_{n=lm} \nu(l)|\nu(m)| \leq \nu(n)\tau_2(n).$$

Assume that (A) holds. By Cauchy's inequality, the first assertion of Lemma 3.2 and Lemma 3.1,

$$\sum_{\psi \in \Psi} \left(|X_4(D^8, \psi)| + \int_{D^4}^{D^8} \frac{|X_4(x, \psi)|}{x} dx \right)^2 \ll \mathcal{P}\mathcal{L}^{-2005}.$$

Thus we conclude

Lemma 3.6. *Assume that (A) holds. The inequality*

$$|X_4(D^8, \psi)| + \int_{D^4}^{D^8} \frac{|X_4(x, \psi)|}{x} dx < \mathcal{L}^{-633} \quad (3.6)$$

holds for all but at most $O(\mathcal{P}\mathcal{L}^{-739})$ characters ψ in Ψ .

We are now in a position to give the definition of Ψ_1 : Let Ψ_1 be the subset of Ψ such that $\psi \in \Psi_1$ if and only if the inequalities (3.4), (3.5) and (3.6) simultaneously hold.

Proposition 2.1 follows from Lemma 3.4, 3.5 and 3.6 immediately.

4. Zeros of $L(s, \psi)L(s, \chi\psi)$ in Ω

In this section we prove Proposition 2.2. We henceforth assume that $\psi(\bmod p) \in \Psi_1$. This assumption will not be repeated in the statements of Lemma 4.1-4.8.

We begin by proving some consequences of the inequalities (3.4)-(3.6).

Lemma 4.1. *Let*

$$\Omega_1 = \{s : 1/2 - (100\mathcal{L})^{-1} \log \mathcal{L} < \sigma < 1 + (100\mathcal{L})^{-1} \log \mathcal{L}, \quad |t - 2\pi t_0| < \mathcal{L}_1 + 5\}.$$

If $s \in \Omega_1$, then

$$|F(s, \psi)| + |G(s, \psi)| \ll \mathcal{L}^{79}.$$

Proof. By the Stieltjes integral we may write

$$F(s, \psi)^{20} = 1 + \int_1^{D^{80}} x^{s_0-s} d\{X_1(x, \psi)\}.$$

For $s \in \Omega_1$ and $1 \leq x \leq D^{80}$ we have

$$|x^{s_0-s}| \ll \mathcal{L}, \quad \left| \frac{d}{dx} (x^{s_0-s}) \right| \ll x^{-1} \mathcal{L}^{406}.$$

Hence, by partial integration,

$$|F(s, \psi)|^{20} \ll 1 + \mathcal{L}^{406} \left(|X_1(D^{80}, \psi)| + \int_1^{D^{80}} \frac{|X_1(x, \psi)|}{x} dx \right).$$

For $G(s, \psi)$ an entirely analogous bound is valid. The result now follows by (3.4). \square

Lemma 4.2. *If $s \in \Omega_1$, then*

$$F(s, \psi)G(s, \psi) = 1 + O(\mathcal{L}^{-227}).$$

Proof. We have

$$F(s, \psi)G(s, \psi) - 1 = \sum_{D^4 < n \leq D^8} \frac{\varsigma(n)\psi(n)}{n^s} = \int_{D^4}^{D^8} x^{s_0-s} d\{X_4(x, \psi)\}.$$

Thus, similar to (4.2), by partial integration we obtain

$$F(s, \psi)G(s, \psi) - 1 \ll \mathcal{L}^{406} \left(|X_4(D^8, \psi)| + \int_{D^4}^{D^8} \frac{|X_4(x, \psi)|}{x} dx \right),$$

the right side being $O(\mathcal{L}^{-227})$ by (3.6). \square

Lemma 4.3. *Let*

$$\Omega_2 = \{s : 1/2 - \mathcal{L}^{-1} < \sigma < 1 + \mathcal{L}^{-1}, \quad |t - 2\pi t_0| < \mathcal{L}_1 + 4\}.$$

If $s \in \Omega_2$, then

$$\frac{F'}{F}(s, \psi) = O(\mathcal{L}).$$

Proof. Assume $|w| \leq (200\mathcal{L})^{-1} \log \mathcal{L}$, so that $s + w \in \Omega_1$. By Lemma 4.1 and 4.2,

$$\mathcal{L}^{-88} \ll |F(s + w, \psi)| \ll \mathcal{L}^{88}.$$

Thus the logarithm

$$\mathfrak{I}(s, w) := \log \frac{F(s + w, \psi)}{F(s, \psi)},$$

which vanishes at $w = 0$, is analytic in w , and it satisfies

$$\Re\{\mathfrak{I}(s, w)\} \ll \log \mathcal{L}.$$

Since

$$\frac{F'}{F}(s, \psi) = \frac{\partial}{\partial w} \mathfrak{I}(s, w)|_{w=0},$$

the result follows by Lemma 4 of [15, Chapter 2]. \square

We proceed to establish an approximate formula for $L(s, \psi)L(s, \chi\psi)$. For this purpose we first introduce a weight $g(x)$ that will find application at various places. Let

$$g(x) = \frac{1}{2\pi i} \int_{(c)} \frac{x^w \omega_1(w)}{w} dw \quad (c > 0)$$

with

$$\omega_1(w) = \exp\{w^2/(4\mathcal{L}^{30})\}.$$

We may write

$$g(x) = \frac{1}{2\pi i} \int_{(c)} \left(\int_0^x y^{w-1} dy \right) \omega_1(w) dw.$$

Since

$$\frac{1}{2\pi i} \int_{(c)} \exp\{(\log y)w + w^2/(4\mathcal{L}^{30})\} dw = \frac{\mathcal{L}^{15}}{\sqrt{\pi}} \exp\{-\mathcal{L}^{30}(\log y)^2\},$$

it follows, by changing the order of integration, that

$$g(x) = \frac{\mathcal{L}^{15}}{\sqrt{\pi}} \int_0^x \exp\{-\mathcal{L}^{30}(\log y)^2\} \frac{dy}{y}.$$

This yields, by substituting $t = \mathcal{L}^{15} \log y$,

$$g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\mathcal{L}^{15} \log x} \exp\{-t^2\} dt. \quad (4.1)$$

Thus the function $g(x)$ is increasing and it satisfies $0 < g(x) < 1$. Further we have

$$g(x) = 1 + O(\exp\{-\mathcal{L}^{30} \log^2 x\}) \quad \text{if } x \geq 1 \quad (4.2)$$

and

$$g(x) = O(\exp\{-\mathcal{L}^{30} \log^2 x\}) \quad \text{if } x \leq 1. \quad (4.3)$$

Note that $\chi\psi$ is a primitive character (mod Dp). Write

$$\tilde{Z}(s, \psi) = Z(s, \psi)Z(s, \chi\psi),$$

so that

$$L(s, \psi)L(s, \chi\psi) = \tilde{Z}(s, \psi)L(1-s, \bar{\psi})L(1-s, \chi\bar{\psi}). \quad (4.4)$$

Recall that $s_0 = 1/2 + 2\pi it_0$. Assume that

$$|\Re(s - s_0)| < 100, \quad |\Im(s - s_0)| < \mathcal{L}_1 + 3.$$

By (2.4) with $\theta = \psi$ and $\theta = \chi\psi$ we have

$$\tilde{Z}(s, \psi) = \chi(-1)\tau(\psi)\tau(\chi\psi)(Dp^2)^{-s}\vartheta(s)^2(1 + O(e^{-\pi t})).$$

This yields, by Stirling's formula,

$$|\tilde{Z}(s, \psi)| = (Dp^2 t_0^2)^{1/2-\sigma}(1 + o(1)) \quad (4.5)$$

and

$$\frac{\tilde{Z}'}{\tilde{Z}}(s, \psi) = -2 \log P + O(\mathcal{L}). \quad (4.6)$$

Lemma 4.4. *Let*

$$\Omega_3 = \{s : 1/2 - \alpha < \sigma < 1 + \alpha, \quad |t - 2\pi t_0| < \mathcal{L}_1 + 3\}.$$

If $s \in \Omega_3$, then

$$L(s, \psi)L(s, \chi\psi) = F(s, \psi) + \tilde{Z}(s, \psi)F(1 - s, \bar{\psi}) + O(\mathcal{L}^{-179}).$$

Proof. By the residue theorem,

$$L(s, \psi)L(s, \chi\psi) = \frac{1}{2\pi i} \left(\int_{(1)} - \int_{(-\sigma-1/2)} \right) L(s+w, \psi)L(s+w, \chi\psi) P^{(9/5)w} \frac{\omega_1(w) dw}{w}.$$

By (4.2) and (4.3),

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(1)} L(s+w, \psi)L(s+w, \chi\psi) P^{(9/5)w} \frac{\omega_1(w) dw}{w} \\ &= F(s, \psi) + \sum_{D^4 < n < P^2} \frac{\nu(n)\psi(n)}{n^s} g\left(\frac{P^{9/5}}{n}\right) + O(\varepsilon) \end{aligned}$$

where

$$\varepsilon = \exp\{-c\mathcal{L}^{10}\}.$$

By (3.5) and partial summation, the second sum on the right side above is

$$= \int_{D^4}^{P^2} g\left(\frac{P^{9/5}}{x}\right) x^{s_0-s} dX_3(x, \psi) \ll \mathcal{L}_1 \left(|X_3(P^2, \psi)| + \int_{D^4}^{P^2} \frac{|X_3(x, \psi)|}{x} dx \right) \ll \mathcal{L}^{-180}.$$

On the other hand, by the functional equation, for $u = -\sigma - 1/2$,

$$L(s+w, \psi)L(s+w, \chi\psi) = \tilde{Z}(s+w, \psi) \sum_n \frac{\nu(n)\bar{\psi}(n)}{n^{1-s-w}}.$$

This sum is split into three sums according to $n \leq D^4$, $D^4 < n \leq P^2$ and $P^2 < n$. The proof is therefore reduced to showing that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(-\sigma-1/2)} \tilde{Z}(s+w, \psi)F(1-s-w, \bar{\psi})P^{(9/5)w} \frac{\omega_1(w) dw}{w} \\ &= -\tilde{Z}(s, \psi)F(1-s, \bar{\psi}) + O(P^{-c}), \end{aligned} \quad (4.7)$$

$$\frac{1}{2\pi i} \int_{(-\sigma-1/2)} \tilde{Z}(s+w, \psi) \left(\sum_{D^4 < n \leq P^2} \frac{\nu(n)\bar{\psi}(n)}{n^{1-s-w}} \right) P^{(9/5)w} \frac{\omega_1(w) dw}{w} \ll \mathcal{L}^{-179} \quad (4.8)$$

and

$$\frac{1}{2\pi i} \int_{(-\sigma-1/2)} \tilde{Z}(s+w, \psi) \left(\sum_{n > P^2} \frac{\nu(n)\bar{\psi}(n)}{n^{1-s-w}} \right) P^{(9/5)w} \frac{\omega_1(w) dw}{w} \ll P^{-c}. \quad (4.9)$$

To prove (4.7) we move the contour of integration to the vertical segments

$$\begin{cases} w = 10 + iv & \text{with } |v| < \mathcal{L}^{20}, \\ w = -\sigma - 1/2 + iv & \text{with } |v| \geq \mathcal{L}^{20}, \end{cases}$$

and to the two connecting horizontal segments

$$w = u \pm i\mathcal{L}^{20} \quad \text{with} \quad -\sigma - 1/2 \leq u \leq 10.$$

By a trivial bound for $\omega_1(w)$, (4.5) and the residue theorem we obtain (4.7).

To prove (4.8) we move the contour of integration to the vertical segments

$$\begin{cases} w = -\alpha + iv & \text{with } |v| < \mathcal{L}^{20}, \\ w = -\sigma - 1/2 + iv & \text{with } |v| \geq \mathcal{L}^{20}, \end{cases}$$

and to the two connecting horizontal segments

$$w = u \pm i\mathcal{L}^{20} \quad \text{with} \quad -\sigma - 1/2 \leq u \leq -\alpha.$$

By a trivial bound for $\omega_1(w)$ and (4.5) we see that the left side of (4.8) is

$$\ll P^{1-2\sigma} \int_{-\mathcal{L}^{20}}^{\mathcal{L}^{20}} \left| \sum_{D^4 < n \leq P^2} \frac{\nu(n)\psi(n)}{n^{s^*+iv}} \right| \frac{dv}{|\alpha + iv|} + \varepsilon$$

with $s^* = 1 + \alpha - \bar{s}$. By partial integration,

$$\sum_{D^4 < n \leq P^2} \frac{\nu(n)\psi(n)}{n^{s^*+iv}} = \int_{D^4}^{P^2} x^{s_0-s^*-iv} dX_3(x, \psi) \ll P^{2\sigma-1} \mathcal{L}_1 \left(|X_3(P^2, \psi)| + \int_{D^4}^{P^2} \frac{|X_3(x, \psi)|}{x} dx \right)$$

for $|v| \leq \mathcal{L}^{20}$. From this and (3.5) we obtain (4.8).

The estimate (4.9) follows by moving the contour of integration to the vertical segments

$$\begin{cases} w = -\sigma - 10 + iv & \text{with } |v| < \mathcal{L}^{20}, \\ w = -\sigma - 1/2 + iv & \text{with } |v| \geq \mathcal{L}^{20}, \end{cases}$$

and to the two connecting horizontal segments

$$w = u \pm i\mathcal{L}^{20} \quad \text{with} \quad -\sigma - 10 \leq u \leq -\sigma - 1/2,$$

and applying (4.5) and trivial bounds for $\omega_1(w)$ and the involved sum. \square

In order to prove Proposition 2.2, it is appropriate to deal with the function

$$\mathcal{A}(s, \psi) = \frac{L(s, \psi)L(s, \chi\psi)}{F(s, \psi)}.$$

By Lemma 4.2, $\mathcal{A}(s, \psi)$ is analytic and it has the same zeros as $L(s, \psi)L(s, \psi\chi)$ in Ω_1 . Further, for $s \in \Omega_1$, we have

$$F(s, \psi)^{-1} \ll \mathcal{L}^{79}$$

by Lemma 4.1 and 4.2. This together with Lemma 4.4 implies that

$$\mathcal{A}(s, \psi) = 1 + \mathcal{B}(s, \psi) + O(\mathcal{L}^{-100}), \quad (4.10)$$

for $s \in \Omega_3$, where

$$\mathcal{B}(s, \psi) = \tilde{Z}(s, \psi) \frac{F(1-s, \bar{\psi})}{F(s, \psi)}.$$

The proof of Proposition 2.2 is reduced to proving three lemmas as follows.

Lemma 4.5. *If*

$$\frac{1}{2} + \alpha^2 < \sigma < 1, \quad |t - 2\pi t_0| < \mathcal{L}_1 + 2,$$

then

$$A(s, \psi) \neq 0.$$

Proof. We discuss in two cases.

Case 1. $1/2 + \mathcal{L}^{-1} \leq \sigma < 1$.

By Lemma 4.2 and trivial estimation,

$$\frac{F(1-s, \bar{\psi})}{F(s, \psi)} \ll D^c.$$

Hence, by (4.5),

$$\mathcal{B}(s, \psi) \ll P^{1/2-\sigma}.$$

The result now follows by (4.10).

Case 2. $1/2 + \alpha^2 \leq \sigma < 1/2 + \mathcal{L}^{-1}$.

Assume $1/2 \leq \sigma' \leq \sigma$. Then both $\sigma' + it$ and $1 - \overline{(\sigma' + it)}$ lie in Ω_2 . Hence, by Lemma 4.3 and (4.6),

$$\begin{aligned} \frac{\mathcal{B}'}{\mathcal{B}}(\sigma' + it, \psi) &= \frac{\tilde{Z}'}{\tilde{Z}}(\sigma' + it, \psi) - \frac{F'}{F}(\sigma' + it, \psi) - \frac{F'}{F}(1 - \sigma' - it, \bar{\psi}) \\ &= -2 \log P + O(\mathcal{L}). \end{aligned} \quad (4.11)$$

Since $|\mathcal{B}(1/2 + it, \psi)| = 1$, it follows that

$$\log |\mathcal{B}(s, \psi)| = \Re \left\{ \int_{1/2}^{\sigma} \frac{\mathcal{B}'}{\mathcal{B}}(\sigma' + it) d\sigma \right\} < (1/2 - \sigma) \log P,$$

Hence, by (4.10),

$$|\mathcal{A}(s, \psi)| > 1 - P^{1/2-\sigma} + O(\mathcal{L}^{-100}) \gg \alpha. \quad \square$$

Lemma 4.6. *Suppose $\rho = \beta + i\gamma$ is a zero of $\mathcal{A}(s, \psi)$ satisfying*

$$\frac{1}{2} \leq \beta < \frac{1}{2} + \alpha^2, \quad |\gamma - 2\pi t_0| < \mathcal{L}_1 + 2.$$

Then $\beta = 1/2$, $\mathcal{A}'(\rho, \psi) \neq 0$ and $\mathcal{A}(1/2 + i\gamma + w, \psi) \neq 0$ if $0 < |w| < \alpha(1 - c'\alpha\mathcal{L})$ where c' is a sufficiently constant.

Proof. It suffices to show that the function $\mathcal{A}(1/2 + i\gamma + w, \psi)$ has exactly one zero inside the circle $|w| = \alpha(1 - c'\alpha\mathcal{L})$, counted with multiplicity. By the Rouché theorem, this can be reduced to proving that

$$|\mathcal{A}(1/2 + i\gamma + w, \psi) - (1 - P^{-2w})| < |1 - P^{-2w}| \quad \text{for} \quad |w| = \alpha(1 - c'\alpha\mathcal{L}),$$

since the function $1 - P^{-2w}$ has exactly one zero inside this circle which is at $w = 0$. In fact, we can prove that

$$\mathcal{A}(1/2 + i\gamma + w, \psi) - (1 - P^{-2w}) \ll \alpha\mathcal{L} \tag{4.12}$$

if $|w| < 2\alpha$, the implied constant being independent of c' , and

$$|1 - P^{-2w}| > 6c'\alpha\mathcal{L} \tag{4.13}$$

if $|w| = \alpha(1 - c'\alpha\mathcal{L})$.

Assume $|w| < 2\alpha$. By (4.10) we have

$$\mathcal{A}(1/2 + i\gamma + w, \psi) - (1 - P^{-2w}) = \mathcal{B}(1/2 + i\gamma + w, \psi) + P^{-2w} + O(\mathcal{L}^{-100}).$$

Noting that both s and $1 - s$ are in Ω_2 if s lies on the segment connecting ρ and $1/2 + i\gamma + w$, by (4.11) we have

$$\begin{aligned} \frac{\mathcal{B}(1/2 + i\gamma + w, \psi)}{\mathcal{B}(\rho, \psi)} &= \exp \left\{ \int_{\rho}^{1/2+i\gamma+w} \frac{\mathcal{B}'}{\mathcal{B}}(s, \psi) ds \right\} = \exp \{ -2w \log P + O(\alpha\mathcal{L}) \} \\ &= P^{-2w} + O(\alpha\mathcal{L}), \end{aligned}$$

Since $\mathcal{B}(\rho, \psi) = -1 + O(\mathcal{L}^{-100})$ by (4.10), the estimate (4.12) follows.

Now assume $|w| = \alpha(1 - c'\alpha\mathcal{L})$. Write

$$w = \alpha(1 - c'\alpha\mathcal{L})(\cos \theta + i \sin \theta).$$

Then

$$\begin{aligned} |P^{-2w}| &= \exp \left\{ -2\pi(1 - c'\alpha\mathcal{L}) \cos \theta \right\}, & \Im\{P^{-2w}\} &= -|P^{-2w}| \sin \left\{ 2\pi(1 - c'\alpha\mathcal{L}) \sin \theta \right\}; \\ &\text{if } \cos \theta > c'\alpha\mathcal{L}, & \text{then } |P^{-2w}| &< 1 - 6c'\alpha\mathcal{L}; \\ &\text{if } \cos \theta < -c'\alpha\mathcal{L}, & \text{then } |P^{-2w}| &> 1 + 6c'\alpha\mathcal{L}; \\ &\text{if } |\cos \theta| \leq c'\alpha\mathcal{L}, & \text{then } |\Im\{P^{-2w}\}| &> 6c'\alpha\mathcal{L}, \end{aligned}$$

since

$$\sqrt{1 - (c'\alpha\mathcal{L})^2} \leq |\sin \theta| \leq 1,$$

so that

$$|\sin \{2\pi(1 - c'\alpha\mathcal{L}) \sin \theta\}| = 2\pi c'\alpha\mathcal{L}(1 + o(1)).$$

In either case (4.16) holds. \square

Lemma 4.5 and 4.6 together imply the assertions (i) and (ii) of Proposition 2.2. It is also proved that the gap between any distinct zeros of $\mathcal{A}(s, \psi)$ in Ω is $> \alpha(1 - c'\alpha\mathcal{L})$. To complete the proof of the gap assertion (iii), it now suffices to prove

Lemma 4.7. *Suppose $\rho = 1/2 + i\gamma$ is a zero of $\mathcal{A}(s, \psi)$ satisfying $|\gamma - 2\pi t_0| < \mathcal{L}_1 + 2$. Then the function $\mathcal{A}(\rho + w, \psi)$ has exactly three zeros inside the circle $|w| = \alpha(1 + c'\alpha\mathcal{L})$, counted with multiplicity.*

Proof. In a way similar to the proof of Lemma 4.6, it is direct to verify that

$$|\mathcal{A}(\rho + w, \psi) - (1 - P^{-2w})| < |1 - P^{-2w}|$$

if $|w| = \alpha(1 + c'\alpha\mathcal{L})$. Hence, the functions $\mathcal{A}(\rho + w, \psi)$ and $1 - P^{-2w}$ have the same number of zeros inside this circle, while the later has exactly three zeros inside the same circle which are at $w = 0$, $w = i\alpha$ and $w = -i\alpha$. \square

We conclude this section by giving a result which is implied in the proof of Proposition 2.2.

Lemma 4.8. *Assume that ρ is a zero of $L(s, \psi)L(s, \chi\psi)$ in Ω . Then we have*

$$\tilde{Z}(\rho, \psi)^{-1} = -G(\rho, \psi)F(1 - \rho, \bar{\psi}) + O(\mathcal{L}^{-100}).$$

Proof. It follows from Lemma 4.4 that

$$F(\rho, \psi) + \tilde{Z}(\rho, \psi)F(1 - \rho, \bar{\psi}) \ll \mathcal{L}^{-179}.$$

The result follows by multiplying both sides by $\tilde{Z}(\rho, \psi)^{-1}G(\rho, \psi)$ and applying Lemma 4.1. \square

5. Some analytic lemmas

Lemma 5.1. *Suppose $\psi(\bmod p) \in \Psi$, $|\sigma - 1/2| \leq \alpha$, $|t - 2\pi t_0| < \mathcal{L}_1 + 2$, $u = 0$ and $|v| < L^{20}$. Then*

$$\frac{Z(s+w, \psi) - Z(s, \psi)(pt_0)^{-w}}{w} \ll \mathcal{L}^{-114} \quad (5.1)$$

$$\frac{Z(s+w, \psi) - Z(s, \psi)(Pt_0)^{-w}}{w} \ll \mathcal{L}^{-68} \quad (5.2)$$

$$\frac{Z(s+w, \chi\psi) - Z(s, \chi\psi)(Dpt_0)^{-w}}{w} \ll \mathcal{L}^{-114} \quad (5.3)$$

and

$$\frac{Z(s+w, \chi\psi) - Z(s, \chi\psi)(DPt_0)^{-w}}{w} \ll \mathcal{L}^{-68} \quad (5.4)$$

Proof. We have

$$\frac{Z(s+w, \psi)}{Z(s, \psi)} = \exp \left\{ \int_0^w \frac{Z'}{Z}(s+w', \psi) dw' \right\}.$$

Assume $|w'| \leq |w|$. By (2.6) and the Stirling formula,

$$\frac{Z'}{Z}(s+w', \psi) = -\log p + \frac{\vartheta'}{\vartheta}(s+w') + O(\varepsilon) = -\log(pt/2\pi) + O(1/t_0).$$

Hence

$$Z(s+w, \psi) = Z(s, \psi)(pt/2\pi)^{-w} + O(|w|/t_0).$$

This yields (5.1) and (5.2) since

$$pt/2\pi = pt_0(1 + O(\mathcal{L}^{-114})), \quad Pt/2\pi = Pt_0(1 + O(\mathcal{L}^{-68})).$$

The proofs of (5.3) and (5.4) are similar. \square

Lemma 5.2. *Let ψ and s be as in Lemma 5.1. Then*

$$\frac{Y(s+\beta_1, \psi)Y(s+\beta_2, \psi)Y(s+\beta_3, \psi)}{Y(s, \psi)} = (pt_0)^{\beta_3} Z(s, \psi)^{-1} (1 + O(\mathcal{L}^{-123})).$$

Proof. The left side is

$$Z(s, \psi)^{-1} \prod_{1 \leq j \leq 3} \left(\frac{Y(s+\beta_j, \psi)}{Y(s, \psi)} \right).$$

By (2.6) and the Stirling formula, for $|w| < 5\alpha$,

$$\frac{Y'}{Y}(s+w, \psi) = -\frac{1}{2} \frac{Z'}{Z}(s+w, \psi) = \frac{1}{2} \log(pt_0) + O(\mathcal{L}^{-114}).$$

Hence, for $1 \leq j \leq 3$,

$$\frac{Y(s + \beta_j, \psi)}{Y(s, \psi)} = \exp\left(\int_0^{\beta_j} \frac{Y'}{Y}(s+w, \psi) dw\right) = (pt_0)^{\beta_j/2} (1 + O(\mathcal{L}^{-123})).$$

The result now follows since

$$\frac{\beta_1 + \beta_2 + \beta_3}{2} = \beta_3. \quad \square$$

Recall that $\vartheta(s)$ and $\omega(s)$ are given by (2.3) and (2.15) respectively. It is known that

$$\vartheta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos(\pi s/2).$$

For $t > 1$ we have

$$\vartheta(1-s) = \vartheta^*(1-s)(1 + O(e^{-\pi t})) \tag{5.5}$$

where

$$\vartheta^*(1-s) = (2\pi)^{-s} \Gamma(s) e(-s/4).$$

Let

$$\Delta_1(x) = \frac{1}{2\pi i} \int_{(3/2)} x^{-s} \vartheta^*(1-s) \omega(s) ds \tag{5.6}$$

and

$$\Delta(x) = \Delta_1(x) e(x). \tag{5.7}$$

Note that

$$\omega(1/2 + 2\pi i x) = \frac{\sqrt{\pi}}{\mathcal{L}_2} \exp\left\{-\left(\frac{\pi(x-t_0)}{\mathcal{L}_2}\right)^2\right\} > 0.$$

Lemma 5.3. *If $x \leq t_0^{1.02}$, then*

$$\Delta(x) = \omega(1/2 + 2\pi i x) (1 + O(\alpha)) + O(\varepsilon); \tag{5.8}$$

if $x > t_0^{1.02}$, then

$$\Delta(x) \ll \exp\{-(10^{-2} \mathcal{L}_2 \log x)^2\} + \exp\{-x^{0.99}/\mathcal{L}_2\}. \tag{5.9}$$

Proof. By the Mellin transform (see [1], Lemma 2) we have

$$\Delta_1(x) = \int_0^\infty \exp\{(s_0 - 1) \log y - \mathcal{L}_2^2 \log^2 y - 2\pi i x y\} dy$$

where the logarithm vanishes at $y = 1$. This yields, by substituting $y = e^u$,

$$\Delta(x) = \int_{-\infty}^\infty \exp\{s_0 u - \mathcal{L}_2^2 u^2 - 2\pi i x (e^u - 1)\} du. \quad (5.10)$$

First assume $x \leq t_0^{1.02}$. We may write

$$\exp\{s_0 w - \mathcal{L}_2^2 w^2 - 2\pi i x (e^w - 1)\} = \exp\{2\pi i (t_0 - x) w - \mathcal{L}_2^2 w^2\} f^*(x, w)$$

with

$$f^*(x, w) = \exp\{(1/2)w - 2\pi i x (e^w - 1 - w)\}.$$

By the relation

$$\int_{-\infty}^\infty \exp\{2\pi i (t_0 - x) u - \mathcal{L}_2^2 u^2\} du = \omega(1/2 + 2\pi i x)$$

and Cauchy's theorem, the proof of (5.8) is reduced to showing that

$$\int_{L_j} \exp\{2\pi i (t_0 - x) w - \mathcal{L}_2^2 w^2\} (f^*(x, w) - 1) dw \ll \alpha \omega(1/2 + 2\pi i x) + \varepsilon \quad (5.11)$$

for $1 \leq j \leq 5$, where L_j denote the segments

$$L_1 = (-\infty, -u^*], \quad L_2 = [-u^*, -u^* + iv^*], \quad L_3 = [-u^* + iv^*, u^* + iv^*],$$

$$L_4 = \{u^* + iv^*, u^*\}, \quad L_5 = [u^*, \infty)$$

with

$$u^* = \mathcal{L}_2^{-1} \mathcal{L}^5, \quad v^* = \frac{\pi(t_0 - x)}{\mathcal{L}_2^2}.$$

If $w \in L_1 \cup L_5$, then

$$\exp\{2\pi i (t_0 - x) w - \mathcal{L}_2^2 w^2\} (f^*(x, w) - 1) \ll \exp\{-\mathcal{L}_2^2 u^2\} (1 + e^{u/2}).$$

Thus the left side of (5.11) is trivially $O(\varepsilon)$ if $j = 1, 5$. By simple estimates,

$$\int_{L_j} |\exp\{2\pi i (t_0 - x) w - \mathcal{L}_2^2 w^2\} dw| \ll \omega(1/2 + 2\pi i x) + \varepsilon$$

if $j = 2, 3, 4$. Since $t_0^{3.06}/\mathcal{L}_2^4 \ll \alpha$, we have

$$f^*(x, w) - 1 \ll |w| + x|w|^2 \ll \alpha$$

for $w \in L_2 \cup L_3 \cup L_4$. These estimates together imply (5.11).

Now assume $x > t_0^{1.02}$. By Cauchy's theorem, the proof of (5.9) is reduced to showing that

$$\int_{L'_j} \exp\{s_0 w - \mathcal{L}_2^2 w^2 - 2\pi i x(e^w - 1)\} dw \ll \exp\{-(10^{-2} \mathcal{L}_2 \log x)^2\} + \exp\{-x^{0.99}/\mathcal{L}_2\} \quad (5.12)$$

where L'_j , $1 \leq j \leq 3$, denote the segments

$$\begin{aligned} L'_1 &= \{-\infty, -10^{-2} \log x\], & L'_2 &= [-10^{-2} \log x, -10^{-2} \log x - i/\mathcal{L}_2], \\ L'_3 &= [-10^{-2} \log x - i/\mathcal{L}_2, \infty - i/\mathcal{L}_2). \end{aligned}$$

We have

$$\Re\{s_0 w - \mathcal{L}_2^2 w^2 - 2\pi i x(e^w - 1)\} = \frac{1}{2}u - \mathcal{L}_2^2(u^2 - v^2) + 2\pi v \left(\frac{x e^u \sin v}{v} - t_0 \right). \quad (5.13)$$

If $w \in L'_1$, then the right side of (5.13) is

$$\leq \frac{1}{2}u - (10^{-2} \mathcal{L}_2 \log x)^2.$$

This yields (5.12) with $j = 1$. For $w \in L'_2 \cup L'_3$ we have

$$x e^u \geq x^{0.99} > 2t_0 \quad \text{and} \quad -1/\mathcal{L}_2 \leq v \leq 0,$$

so that

$$2\pi \left(\frac{x e^u \sin v}{v} - t_0 \right) > x^{0.99}.$$

If $w \in L'_2$, then $(\mathcal{L}_2 u)^2 = (10^{-2} \mathcal{L}_2 \log x)^2$, so the right side of (5.13) is

$$\leq -(10^{-2} \mathcal{L}_2 \log x)^2 + O(1).$$

This yields (5.12) with $j = 2$. If $w \in L'_3$, then $v = -1/\mathcal{L}_2$, and the right side of (5.13) is

$$< 1 + \frac{1}{2}u - (\mathcal{L}_2 u)^2 - x^{0.99}/\mathcal{L}_2.$$

This yields (5.12) with $j = 3$. \square

As a consequence of Lemma 5.3, the Mellin transform

$$\delta(s) := \int_0^\infty \Delta(x) x^{s-1} dx \quad (5.14)$$

is analytic for $\sigma > 0$.

Lemma 5.4. (i). If $1/2 \leq \sigma \leq 2$, then

$$\delta(s) \ll \mathcal{L}^c |s|^{-2}.$$

(ii). If $|s - 1| < 10\alpha$, then

$$\delta(s) = 1 + O(\alpha \log \mathcal{L}).$$

Proof. (i). Using partial integration twice we obtain

$$\delta(s) = \frac{1}{s(s+1)} \int_0^\infty \Delta''(x) x^{s+1} ds.$$

By (5.10) we have

$$\Delta''(x) = -4\pi^2 \int_{-\infty}^\infty (e^u - 1)^2 \exp\{s_0 u - B^2 u^2 - 2\pi i x (e^u - 1)\} du.$$

Thus some upper bounds for $\Delta''(x)$ analogous to Lemma 5.3 can be obtained, and (i) follows.

(ii). Assume $|s - 1| < 10\alpha$. By Lemma 5.3, on the right side of (5.14), the integral on the part $|x - t_0| \geq \mathcal{L}_1$ contributes $O(\varepsilon)$. For $|x - t_0| < \mathcal{L}_1$ we have

$$x^{s-1} = 1 + O(\alpha \log \mathcal{L}).$$

Since

$$\int_0^\infty \omega(1/2 + 2\pi i x) dx = 1 + O(\varepsilon),$$

(ii) follows. \square

Throughout the rest of this paper we assume that (A) holds. This assumption will not be repeated in the statements of the lemmas and propositions in the sequel.

The next two lemmas are weaker forms of the Deuring-Heilbronn Phenomenon.

Lemma 5.5. *The function $L(s, \chi)$ has a simple real zero $\tilde{\rho}$ such that*

$$1 - \tilde{\rho} = O(\mathcal{L}^{-2022}) \tag{5.15}$$

and $L(s, \chi)$ has no other zeros in the region

$$\sigma > 1 - 2\mathcal{L}^{-1}, \quad |t| < 2D.$$

Lemma 5.6. For any primitive character $\theta(\bmod r)$ with $r < T$ and $\theta \neq \chi$,

$$\sum_{p \sim P} \theta(p) p^{1+it} \ll \mathcal{P} \exp\{-\mathcal{L}^{9/2}\}$$

if $|t| \leq D$.

Lemma 5.7. We have

$$L'(1, \chi) \gg \frac{D}{\varphi(D)}.$$

Proof. The right side of the equality

$$\frac{1}{2\pi i} \int_{(1)} \zeta(1+s) L(1+s, \chi) \frac{D^{4s} \omega_1(s)}{s} ds = \sum_n \frac{\nu(n)}{n} g\left(\frac{D^4}{n}\right).$$

is

$$\gg \sum_{n|D} \frac{1}{n} \gg \frac{D}{\varphi(D)},$$

since $\nu(n) = 1$ if $n|D$, while the left side is, by moving the line of integration to the left appropriately and applying (A), equal to $L'(1, \chi) + o(1)$. \square

Lemma 5.8. If

$$\alpha \leq |s-1| \leq 10\alpha,$$

then

$$L(s, \chi) = L'(1, \chi)(s-1) + O(\alpha_2)$$

where

$$\alpha_2 = \mathcal{L}^{-15}.$$

Proof. This follows from the relation

$$L(s, \chi) = L(1, \chi) + L'(1, \chi)(s-1) + \int_1^s (s-w) L''(w, \chi) dw,$$

(A) and a simple bound for $L''(w, \chi)$. \square

Lemma 5.9. Suppose $\psi \in \Psi_1$, $|\sigma - 1/2| \leq \alpha$, $|t - 2\pi t_0| \leq \mathcal{L}_1 + 10$ and $|s - \rho| \gg \alpha$ for any zero ρ of $L(s, \psi)$. Then

$$\frac{L(s + \beta_1, \psi)}{L(s, \psi)} \ll \log P.$$

Proof. It is known that

$$\frac{L'}{L}(s', \psi) = \sum_{|\rho-s'| < 1} \frac{1}{s' - \rho} + O(1/\alpha) \tag{5.16}$$

for $|\Re\{s'\} - 1/2| \leq \alpha$ and $|\Im\{s' - 2\pi t_0\}| < \mathcal{L}_1 + 10$, where ρ runs through the zeros of $L(s', \psi)$.

We can assume $L(s + \beta_1, \psi) \neq 0$. Suppose $\sigma \geq 1/2$. By (5.16),

$$-\Re\left\{\frac{L'}{L}(u + it + \beta_1, \psi)\right\} < O(1/\alpha) \quad \text{for } \sigma < u < 1/2 + \alpha,$$

so that

$$\log \frac{|L(s + \beta_1, \psi)|}{|L(1/2 + \alpha + it + \beta_1, \psi)|} < O(1).$$

By (9.1) and the condition $|s - \rho| \gg \alpha$ for any ρ ,

$$\Re\left\{\frac{L'}{L}(u + it, \psi)\right\} = O(1/\alpha) \quad \text{for } \sigma < u < 1/2 + \alpha,$$

so that

$$\log \frac{|L(1/2 + \alpha + it, \psi)|}{|L(s, \psi)|} < O(1).$$

Further, by (5.16) and Proposition 2.2 (iii),

$$\left|\frac{L'}{L}(1/2 + \alpha + it', \psi)\right| < \frac{1}{\alpha} \sum_{k < 1/\alpha} \frac{1}{k} + O(1/\alpha) < \frac{\log \log P}{\alpha} + O(1/\alpha)$$

for $t < t' < t + |\beta_1|$, so that

$$\log \frac{|L(1/2 + \alpha + it + \beta_1, \psi)|}{|L(1/2 + \alpha + it, \psi)|} < \log \log P + O(1).$$

Combining these estimates we obtain the result. In the case $\sigma < 1/2$ the proof is analogous. \square

6. Approximate formula for $L(s, \psi)$

Write

$$g^*(y) = \begin{cases} g(y) & \text{if } y > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$P_4 = PT^{-2}t_0 \quad \text{with } T = \exp\{\mathcal{L}^{1.1}\}.$$

Lemma 6.1. *Suppose $\psi \pmod{p} \in \Psi$, $|\sigma - 1/2| < 2\alpha$ and $|t - 2\pi t_0| < \mathcal{L}_1 + 2$. Then*

$$L(s, \psi) = K(s, \psi) + Z(s, \psi)N(1 - s, \bar{\psi}) + O(E_1(s, \psi)),$$

where

$$K(s, \psi) = \sum_n \frac{\psi(n)}{n^s} g^* \left(\frac{P_4}{n} \right)$$

$$N(1-s, \bar{\psi}) = \sum_n \frac{\bar{\psi}(n)}{n^{1-s}} g^* \left(\frac{T^2}{n} \right)$$

and where

$$E_1(s, \psi) = \mathcal{L}^{-68} \int_{-\mathcal{L}^{20}}^{\mathcal{L}^{20}} \left| \sum_{n < T^3} \frac{\psi(n)}{n^{s+iv}} \right| \omega_1(iv) dv + \varepsilon.$$

Proof. By (4.3) we have

$$\frac{1}{2\pi i} \int_{(1)} L(s+w, \psi) \frac{P_4^w \omega_1(w)}{w} dw = K(s, \psi) + O(\varepsilon).$$

The left side above is, by moving the line of integration to $u = -1$, equal to

$$L(s, \psi) + \frac{1}{2\pi i} \int_{(-1)} L(s+w, \psi) \frac{P_4^w \omega_1(w)}{w} dw.$$

It therefore suffices to show that

$$\frac{1}{2\pi i} \int_{(-1)} L(s+w, \psi) \frac{P_4^w \omega_1(w)}{w} dw = -Z(s, \psi) N(1-s, \bar{\psi}) + O(E_1(s, \psi)). \quad (6.1)$$

For $u = -1$ we have, by the functional equation (2.2) with $\theta = \psi$,

$$L(s+w, \psi) = Z(s+w, \psi) \left(\sum_{n < T^3} \frac{\bar{\psi}(n)}{n^{1-s-w}} + \sum_{n \geq T^3} \frac{\bar{\psi}(n)}{n^{1-s-w}} \right)$$

We first show that

$$\int_{(-1)} Z(s+w, \psi) \left(\sum_{n \geq T^3} \frac{\bar{\psi}(n)}{n^{1-s-w}} \right) \frac{P_4^w \omega_1(w)}{w} dw \ll \varepsilon. \quad (6.2)$$

We move the contour of integration in (6.2) to the vertical segments

$$u = -1, \quad |v| > \mathcal{L}^{20}$$

and

$$u = -\mathcal{L}^9, \quad |v| \leq \mathcal{L}^{20}$$

with the horizontal connecting segments

$$-\mathcal{L}^9 \leq u \leq -1 \quad |v| = \mathcal{L}^{20}.$$

By a trivial bound for $\omega_1(w)$ and simple estimates, the integrals on the segments $u = -1$, $|v| > \mathcal{L}^{20}$ contribute $\ll \varepsilon$. On the other hand, in the case $-\mathcal{L}^9 \leq u \leq -1$, $|v| \leq \mathcal{L}^{20}$, by (4.) we have

$$Z(s+w, \psi)P_4^w \ll (2T^2)^{-u}$$

and

$$\sum_{n \geq T^3} \frac{\bar{\psi}(n)}{n^{1-s-w}} \ll T^{3u+1/2},$$

whence (6.2) follows. The proof of (6.1) is therefore reduced to showing that

$$\begin{aligned} \frac{1}{2\pi i} \int_{-1-i\mathcal{L}^{20}}^{-1+i\mathcal{L}^{20}} Z(s+w, \psi) \left(\sum_{n < T^3} \frac{\bar{\psi}(n)}{n^{1-s-w}} \right) \frac{P_4^w \omega_1(w) dw}{w} \\ = -Z(s, \psi)N(1-s, \bar{\psi}) + O(E_1(s, \psi)). \end{aligned} \quad (6.3)$$

Note that $Pt_0/P_4 = T^2$. The left side of (6.3) is equal to $I' + I''$ with

$$I' = Z(s, \psi) \cdot \frac{1}{2\pi i} \int_{-1-i\mathcal{L}^{20}}^{-1+i\mathcal{L}^{20}} \left(\sum_{n < T^3} \frac{\bar{\psi}(n)}{n^{1-s-w}} \right) \frac{T^{-2w} \omega_1(w) dw}{w}, \quad (6.4)$$

$$I'' = \frac{1}{2\pi i} \int_{(-1)} (Z(s+w, \psi) - Z(s, \psi)(Pt_0)^{-w}) \left(\sum_{n < T^2} \frac{\bar{\psi}(n)}{n^{1-s-w}} \right) \frac{P_4^w \omega_1(w) dw}{w}. \quad (6.5)$$

Replacing the segment $u = -1$, $|v| \leq \mathcal{L}^{20}$ by $u = -1$ and using the change of variable $w \rightarrow -w$, we obtain

$$I' = -Z(s, \psi)N(1-s, \bar{\psi}) + O(\varepsilon).$$

On the other hand, moving the segment $u = -1$, $|v| \leq \mathcal{L}^{20}$ to $u = 0$, $|v| \leq \mathcal{L}^{20}$ and applying Lemma 5.1, we find the the right side of (6.5) is $\ll E_1(s, \psi)$. \square

7. Mean-value formula I

Let $\mathcal{N}(d)$ denote the set of positive integers such that $h \in \mathcal{N}(d)$ if and only if every prime factor of h divides d (note that $1 \in \mathcal{N}(d)$ for every d and $\mathcal{N}(1) = \{1\}$). Assume $1 \leq j \leq 3$ in what follows. Write

$$\tilde{\kappa}(d; m, s) = \sum_{\substack{h \in \mathcal{N}(d) \\ (h, m) = 1}} \frac{\kappa(dh)}{h^s},$$

and

$$\lambda(m, s) = \prod_{q|m} \frac{(1 - q^{-s-\beta_1})(1 - q^{-s-\beta_2})(1 - q^{-s-\beta_3})}{1 - q^{-s}}.$$

For notational simplicity we write

$$\tilde{\kappa}(d; m, 1 - \beta_j) = \tilde{\kappa}_{0j}(d; m), \quad \lambda(m, 1 - \beta_j) = \lambda_{0j}(m).$$

Let

$$\xi_{0j}(n; d, r) = \tilde{\lambda}_{0j}(n, dr) \sum_{\substack{n=d_1k \\ (k,r)=1}} \frac{\tilde{\kappa}_{0j}(d_1; drk)\mu(k)k^{1-\beta_j}}{\varphi(k)}$$

with

$$\tilde{\lambda}_{0j}(n, dr) = \prod_{\substack{q|n \\ (q,dr)=1}} \lambda_{0j}(q),$$

For $\psi(\bmod p) \in \Psi$ write

$$\mathcal{C}(s, \psi) = -i(pt_0)^{\beta_3} Z(s, \psi)^{-1} \frac{L(s + \beta_1, \psi)L(s + \beta_2, \psi)L(s + \beta_3, \psi)}{L(s, \psi)}. \quad (7.1)$$

Let $\mathbf{a} = \{a(n)\}$ denote a sequence of complex numbers satisfying

$$a(n) \ll 1, \quad a(n) = 0 \quad \text{if } n \geq PT^{-2}. \quad (7.2)$$

Write

$$\bar{\mathbf{a}} = \{\bar{a}(n)\}, \quad \bar{a}(n) = \overline{a(n)},$$

$$A(\mathbf{a}; s, \psi) = \sum_n \frac{a(n)\psi(n)}{n^s}, \quad A(\mathbf{a}; 1 - s, \bar{\psi}) = \sum_n \frac{a(n)\bar{\psi}(n)}{n^{1-s}}.$$

Let $\mathcal{J}(z)$ denote the segment $[s_0 + z - i\mathcal{L}_1, s_0 + z + i\mathcal{L}_1]$.

The goal of this section is to prove

Proposition 7.1. *Assume \mathbf{a}_1 and \mathbf{a}_2 satisfy (7.2). We have*

$$\begin{aligned} \Theta_1(\mathbf{a}_1, \mathbf{a}_2) &:= \sum_{\psi \in \Psi_1} \frac{1}{2\pi i} \int_{\mathcal{J}(1)} \mathcal{C}(s, \psi) A(\mathbf{a}_1; s, \psi) A(\mathbf{a}_2; 1 - s, \bar{\psi}) \omega(s) ds \\ &= \frac{1}{\alpha} \left(\frac{1}{2} S_1(\mathbf{a}_1, \mathbf{a}_2) + 2S_2(\mathbf{a}_1, \mathbf{a}_2) + \frac{3}{2} S_3(\mathbf{a}_1, \mathbf{a}_2) \right) \mathcal{P} \\ &\quad + O(E(\mathbf{a}_1, \mathbf{a}_2)) + o(\mathcal{P}), \end{aligned}$$

where

$$\mathcal{P} = \sum_{p \sim P} p,$$

$$S_j(\mathbf{a}_1, \mathbf{a}_2) = \sum_d \sum_r \frac{|\mu(r)|\lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{a_1(drm)}{m^{1-\beta_j}} \right) \left(\sum_n \frac{a_2(drn)\xi_{0j}(n; d, r)}{n} \right),$$

and

$$E(\mathbf{a}_1, \mathbf{a}_2) = \mathcal{P}\mathcal{L}^2 \sum_{1 \leq j \leq 3} |S_j(\mathbf{a}_1, \mathbf{a}_2)|.$$

In this and the next two sections we assume that $1 \leq j \leq 3$.

Proof of Proposition 7.1: Initial steps

For notational simplicity we write Θ_1 for $\Theta_1(\mathbf{a}_1, \mathbf{a}_2)$. Note that for any $\psi \in \Psi$, $\mathcal{C}(s, \psi)$ is analytic if $\sigma > 1$ and $|t - 2\pi t_0| \leq \mathcal{L}_1$. We need to show that the sum over Ψ_1 in the expression for Θ_1 can be extended to the sum over Ψ with an acceptable error. Namely we prove

$$\sum_{\psi \in \Psi_2} \int_{\mathcal{J}(1)} \mathcal{C}(s, \psi) A(\mathbf{a}_1; s, \psi) A(\mathbf{a}_2; 1 - s, \bar{\psi}) \omega(s) ds = o(\mathcal{P}). \quad (7.3)$$

Here Proposition 2.1 is crucial.

Let $\kappa(n)$ be given by

$$\sum_n \frac{\kappa(n)}{n^s} = \frac{\zeta(s + \beta_1)\zeta(s + \beta_2)\zeta(s + \beta_3)}{\zeta(s)},$$

and regard a_1 as an arithmetic function. For $\sigma > 1$

$$\frac{L(s + \beta_1, \psi)L(s + \beta_2, \psi)L(s + \beta_3, \psi)}{L(s, \psi)} A(\mathbf{a}_1; s, \psi) = \sum_m \frac{(\kappa * a_1)(m)\psi(m)}{m^s}.$$

Thus, for $\psi \pmod{p} \in \Psi$ and $s \in \mathcal{J}(1)$,

$$\mathcal{C}(s, \psi) A(\mathbf{a}_1; s, \psi) = -i(pt_0)^{\beta_3} Z(s, \psi)^{-1} \sum_m \frac{(\kappa * a_1)(m)\psi(m)}{m^s}.$$

The sum over m is split into two sums according to $m < P^2$ and $m \geq P^2$. To handle the second one we appeal to the trivial bounds

$$\sum_{m \geq P^2} \frac{(\kappa * a_1)(m)\psi(m)}{m^s} \ll P^{2(1-\sigma)} \mathcal{L}^c, \quad A(\mathbf{a}_2; 1 - s, \bar{\psi}) \ll (PT^{-2})^\sigma$$

for $\sigma \geq 3/2$, and

$$Z(s, \psi)^{-1} \ll (pt_0)^{\sigma-1/2}$$

for $3/2 \leq \sigma \leq \mathcal{L}^9$, $|t - 2\pi t_0| \leq \mathcal{L}_1$. Thus, moving the segment to $\mathcal{J}(\mathcal{L}^9)$ and applying the simple estimate

$$\int_{\mathcal{J}(z)} |\omega(s) ds| \ll 1, \quad |z| \leq \mathcal{L}^9, \quad (7.4)$$

we obtain

$$\int_{\mathcal{J}(1)} Z(s, \psi)^{-1} \left(\sum_{m \geq P^2} \frac{(\kappa * a_1)(m) \psi(m)}{m^s} \right) A(\mathbf{a}_2; 1 - s, \bar{\psi}) \omega(s) ds \ll \varepsilon.$$

To handle the sum over $m < P^2$ we move the path of integration to $\mathcal{J}(0)$. Hence

$$\begin{aligned} & \left| \int_{\mathcal{J}(1)} \tilde{\mathcal{C}}(s, \psi) A(\mathbf{a}_1; s, \psi) A(\mathbf{a}_2; 1 - s, \bar{\psi})(s, \psi) \omega(s) ds \right| \\ & \leq \int_{\mathcal{J}(0)} \left| \sum_{m < P^2} \frac{(\kappa * a_1)(m) \psi(m)}{m^s} \right| |A(\mathbf{a}_2; 1 - s, \bar{\psi}) \omega(s)| ds + O(\varepsilon). \end{aligned}$$

By (7.4), the proof of (7.3) is reduced to showing that

$$\sum_{\psi \in \Psi_2} \left| \sum_{m < P^2} \frac{(\kappa * a_1)(m) \psi(m)}{m^s} \right| |A(\mathbf{a}_2; 1 - s, \bar{\psi})| = o(\mathcal{P}), \quad \sigma = 1/2. \quad (7.5)$$

Note that $(\kappa * a_1)(m) = O(\tau_5(m))$. The left side above is, by Cauchy's inequality,

$$\begin{aligned} & \leq \left(\sum_{\psi \in \Psi} \left| \sum_{m < P^2} \frac{(\kappa * a_1)(m) \psi(m)}{m^s} \right|^2 \right)^{1/2} \left(\sum_{\psi \in \Psi} |A(\mathbf{a}_2; 1 - s, \bar{\psi})|^4 \right)^{1/4} \left(\sum_{\psi \in \Psi_2} 1 \right)^{1/4} \\ & \ll \left(P^2 \sum_{m < P^2} \frac{\tau_5(m)^2}{m} \right)^{1/2} \left(P^2 \sum_{m < P^2} \frac{\tau_2(m)^2}{m} \right)^{1/4} \left(\sum_{\psi \in \Psi_2} 1 \right)^{1/4}. \end{aligned}$$

This yields (7.5) by Proposition 2.1 and (2.9).

By (7.3) we may write

$$\Theta_1 = -i \sum_{p \sim P} (pt_0)^{\beta_3} \sum_{\psi(\bmod p)}^* I_1(\psi) + o(\mathcal{P}) \quad (7.6)$$

where

$$I_1(\psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(1)} Z(s, \psi)^{-1} \left(\sum_m \frac{(\kappa * a_1)(m) \psi(m)}{m^s} \right) A(\mathbf{a}_2; 1 - s, \bar{\psi}) \omega(s) ds.$$

Assume $\psi(\bmod p) \in \Psi$. By (2.4), in the integral $I_1(\psi)$, the factor $Z(s, \psi)^{-1}$ can be replaced by

$$\tau(\bar{\psi}) p^{s-1} \vartheta^*(1-s),$$

and then, by a trivial bound for $\vartheta^*(1-s)\omega(s)$ on $\sigma = 3/2$, the segment $\mathcal{J}(1)$ can be replaced by the line $\sigma = 3/2$, with negligible errors. Thus, integration term-by-term gives

$$I_1(\psi) = \frac{\tau(\bar{\psi})}{p} \sum_m \sum_n \frac{(\kappa * a_1)(m) a_2(n) \psi(m) \bar{\psi}(n)}{n} \Delta_1 \left(\frac{m}{pn} \right) + O(\varepsilon).$$

This yields

$$\sum_{\psi \pmod{p}}^* I_1(\psi) = \frac{1}{p} \sum_m \sum_n \frac{(\kappa * a_1)(m)a_2(n)}{n} \Delta_1\left(\frac{m}{pn}\right) \left(\sum_{\psi \pmod{p}}^* \tau(\bar{\psi})\psi(m)\bar{\psi}(n) \right) + O(\varepsilon).$$

Note that $(n, p) = 1$ if $n < PT^{-2}$. On substituting $d = (m, n)$, $m = dl$, $n = dk$, we find that the main term on the right side above is

$$\frac{1}{p} \sum_d \frac{1}{d} \sum_l \sum_{(k,l)=1} \frac{(\kappa * a_1)(dl)a_2(dk)}{k} \left(\sum_{\psi \pmod{p}}^* \tau(\bar{\psi})\psi(l)\bar{\psi}(k) \right) \Delta_1\left(\frac{l}{pk}\right).$$

Since $\tau(\psi_p^0) = -1$, for $(kl, p) = 1$ we have

$$\sum_{\psi \pmod{p}}^* \tau(\bar{\psi})\psi(l)\bar{\psi}(k) = pe\left(\frac{l\bar{k}}{p}\right) + O(1)$$

with $\bar{k}k \equiv 1 \pmod{p}$. Hence, by Lemma 5.4,

$$\sum_{\psi \pmod{p}}^* I(\psi) = \sum_d \frac{1}{d} \sum_{(l,p)=1} \sum_{(k,l)=1} \frac{(\kappa * a_1)(dl)b(dk)}{k} e\left(\frac{l\bar{k}}{p}\right) \Delta_1\left(\frac{l}{pk}\right) + O(PT^{-c}). \quad (7.7)$$

By trivial estimation, this remains valid if the constraint $(l, p) = 1$ is removed. Further, by the relation

$$\frac{\bar{k}}{p} \equiv -\frac{\bar{p}}{k} + \frac{1}{pk} \pmod{1}$$

we have

$$e\left(\frac{l\bar{k}}{p}\right) \Delta_1\left(\frac{l}{pk}\right) = e\left(-\frac{l\bar{p}}{k}\right) \Delta\left(\frac{l}{pk}\right)$$

Thus the right side of (7.7) is

$$\sum_d \frac{1}{d} \sum_l \sum_{(k,l)=1} \frac{(\kappa * a_1)(dl)b(dk)}{k} e\left(-\frac{l\bar{p}}{k}\right) \Delta\left(\frac{l}{pk}\right) + O(PT^{-c}). \quad (7.8)$$

For $(l, k) = 1$ we have

$$e\left(\frac{-l\bar{p}}{k}\right) = \frac{\mu(k)}{\varphi(k)} + \frac{1}{\varphi(k)} \sum'_{\theta \pmod{k}} \tau(\bar{\theta})\theta(l)\bar{\theta}(-p).$$

Inserting this into (7.8) we deduce that

$$\sum_{\psi \pmod{p}}^* I(\psi) = \mathcal{T}_{11}(p) + \mathcal{T}_{12}(p) + O(PT^{-c}) \quad (7.9)$$

where

$$\mathcal{T}_{11}(p) = \sum_d \frac{1}{d} \sum_l \sum_{(k,l)=1} \frac{(\kappa * a_1)(dl) a_2(dk) \mu(k)}{k\varphi(k)} \Delta\left(\frac{l}{pk}\right)$$

and

$$\mathcal{T}_{12}(p) = \sum_d \frac{1}{d} \sum_l \sum_{(k,l)=1} \frac{(\kappa * a_1)(dl) a_2(dk)}{k\varphi(k)} \left(\sum'_{\theta \pmod{k}} \tau(\bar{\theta}) \theta(l) \bar{\theta}(-p) \right) \Delta\left(\frac{l}{pk}\right).$$

Note that $(pt_0)^{\beta_3} = -1 + O(\alpha_1)$. By (7.6) and (7.9), the proof of Proposition 7.1 is reduced to showing that

$$\begin{aligned} i\mathcal{T}_{11}(p) &= \frac{p}{\alpha} \left(\frac{1}{2} S_1(\mathbf{a}_1, \mathbf{a}_2) + 2S_2(\mathbf{a}_1, \mathbf{a}_2) + \frac{3}{2} S_3(\mathbf{a}_1, \mathbf{a}_2) \right) \\ &\quad + O\left(P\mathcal{L}^2 \sum_{1 \leq j \leq 3} |S_j(\mathbf{a}_1, \mathbf{a}_2)| \right) + o(P) \end{aligned} \quad (7.10)$$

for $p \sim P$, and

$$\sum_{p \sim P} p^{\beta_3} \mathcal{T}_{12}(p) = o(P). \quad (7.11)$$

Proof of Proposition 7.1: The error term

In this subsection we prove (7.11).

Changing the order of summation gives

$$\begin{aligned} \sum_{p \sim P} p^{\beta_3} \mathcal{T}_{12}(p) &= \sum_d \frac{1}{d} \sum_k \frac{a_2(dk)}{k\varphi(k)} \sum'_{\theta \pmod{k}} \tau(\bar{\theta}) \sum_l (\kappa * a_1)(dl) \theta(l) \\ &\quad \times \sum_{p \sim P} p^{\beta_3} \bar{\theta}(-p) \Delta\left(\frac{l}{pk}\right). \end{aligned} \quad (7.12)$$

If $k = hr$, $r > 1$ and $\theta \pmod{k}$ is induced by a primitive character $\theta^* \pmod{r}$, then $|\tau(\bar{\theta})| \leq r^{1/2}$. Thus, the inner sum over the non-principal $\theta \pmod{k}$ above is

$$\ll \sum_{\substack{hr=k \\ r>1}} r^{1/2} \sum_{\theta \pmod{r}}^* |\mathcal{S}(r, h, d; \theta)|,$$

where

$$\mathcal{S}(r, h, d; \theta) = \sum_{(l,h)=1} (\kappa * a_1)(dl) \theta(l) \sum_{p \sim P} p^{\beta_3} \bar{\theta}(p) \Delta\left(\frac{l}{phr}\right).$$

Inserting this into (7.12) we obtain

$$\sum_{p \sim P} p^{\beta_3} \mathcal{T}_{12}(p) \ll \sum_{\{d,h,r\}}' \frac{1}{dh\varphi(hr)\sqrt{r}} \sum_{\theta \pmod{r}}^* |\mathcal{S}(r, h, d; \theta)|, \quad (7.13)$$

where $\sum'_{\{d,h,r\}}$ denotes a sum over the triples $\{d, h, r\}$ satisfying $dhr < P_1$ and $r > 1$. By Lemma 5.3, for $dhr < P_1$, the terms in $\mathcal{S}(r, h, d; \theta)$ with $l > P^2$ make a negligible error. Also, by the Mellin transform and Lemma 5.4 (i),

$$\sum_{p \sim P} p^{\beta_3} \bar{\theta}(p) \Delta\left(\frac{l}{phr}\right) \ll \mathcal{L}^c h r l^{-1} \int_{-\infty}^{\infty} \left| \sum_{p \sim P} p^{1+it+\beta_3} \bar{\theta}(p) \right| \frac{dt}{1+t^2} \quad (7.14)$$

for $l \leq P^2$.

Assume $1 < r < D$ and θ is a primitive character (mod r). By Lemma 5.6, the right side of (7.14) is

$$\ll \frac{hr}{l} P^2 D^{-c}.$$

Hence

$$\mathcal{S}(r, h, d; \theta) \ll \tau_5(d) h r P^2 D^{-c}.$$

Thus, on the right side of (7.13), the total contribution from the terms with $1 < r < D$ is $O(P^2 D^{-c})$. The proof of (7.11) is therefore reduced to showing that

$$\frac{1}{R^{3/2}} \sum_{R \leq r < 2R} \sum_{\theta \pmod{r}}^* |\mathcal{S}(r, h, d; \theta)| \ll \tau_5(d) h P^2 D^{-c} \quad (7.15)$$

for

$$dh < P_1 \quad \text{and} \quad D \leq R < (dh)^{-1} P_1$$

which are henceforth assumed .

Let $\mathfrak{J}(y)$ denote the interval

$$[(1/3)Pt_0y, 4Pt_0y].$$

By Lemma 5.1, for $R \leq r < 2R$, the terms with $l \notin \mathfrak{J}(Rh)$ in $\mathcal{S}(r, h, d; \theta)$ make a negligible contribution. Thus, it suffices to prove (7.15) with $\mathcal{S}(r, h, d; \theta)$ replaced by

$$\mathcal{S}^*(R, r, h, d; \theta) := \sum_{\substack{l \in \mathfrak{J}(Rh) \\ (l,h)=1}} (\kappa * a_1)(dl) \theta(l) \sum_{p \simeq P} \bar{\theta}(p) p^{\beta_3} \Delta\left(\frac{l}{phr}\right).$$

By the Mellin transform and Lemma 5.4 (i),

$$\mathcal{S}^*(R, r, h, d; \theta) \ll \mathcal{L}^c \int_{-\infty}^{\infty} \left| \sum_{\substack{l \in \mathfrak{J}(Rh) \\ (l,h)=1}} (\kappa * a_1)(dl) \theta(l) l^{-1-it} \right| \left| \sum_{p \simeq P} \bar{\theta}(p) p^{1+it+\beta_3} \right| \frac{dt}{1+t^2}.$$

For $\sigma = 1$, by the large sieve inequality we have

$$\sum_{R \leq r < 2R} \sum_{\theta \pmod{r}}^* \left| \sum_{\substack{l \in \mathcal{J}(Rh) \\ (l,h)=1}} (\kappa * a_1)(dl) \theta(l) l^{-s} \right|^2 \ll \tau_5(d)^2 \mathcal{L}^c(R^2 + PRht_0)(PRht_0)^{-1} \ll \tau_5(d)^2 \mathcal{L}^c$$

and

$$\sum_{R \leq r < 2R} \sum_{\theta \pmod{r}}^* \left| \sum_{p \sim P} \bar{\theta}(p) p^s \right|^2 \ll (R^2 + P) P^3.$$

It follows by Cauchy's inequality that

$$\frac{1}{R^{3/2}} \sum_{R \leq r < 2R} \sum_{\theta \pmod{r}}^* |\mathcal{S}^*(R, r, h, d; \theta)| \ll \tau_5(d) h \mathcal{L}^c(R^{1/2} P^{3/2} + R^{-1/2} P^2) \ll \tau_5(d) h P^2 D^{-c}.$$

This yields (7.15).

Proof of Proposition 7.1: The main term

In this subsection we prove (7.10).

Assume $p \sim P$. We may write

$$\mathcal{T}_{11}(p) = \sum_d \frac{1}{d} \sum_k \frac{a_2(dk) \mu(k)}{k \varphi(k)} \sum_{(l,k)=1} (\kappa * a_1)(dl) \Delta\left(\frac{l}{pk}\right). \quad (7.16)$$

Since

$$(\kappa * a_1)(dl) = \sum_{m_1 m_2 = dl} \kappa(m_1) a_1(m_2) = \sum_{d=d_1 d_2} \sum_{\substack{m_1 m_2 = dl \\ (m_1, d) = d_1}} \kappa(m_1) a_1(m_2),$$

it follows, by substituting $m_1 = d_1 l_1$, that

$$(\kappa * a_1)(dl) = \sum_{d=d_1 d_2} \sum_{\substack{l=l_1 l_2 \\ (l_1, d_2)=1}} \kappa(d_1 l_1) a_1(d_2 l_2). \quad (7.17)$$

Hence,

$$\sum_{(l,k)=1} (\kappa * a_1)(dl) \Delta\left(\frac{l}{pk}\right) = \sum_{d=d_1 d_2} \sum_{(l_2, k)=1} a_1(d_2 l_2) \sum_{(l_1, k d_2)=1} \kappa(d_1 l_1) \Delta\left(\frac{l_1 l_2}{pk}\right).$$

Inserting this into (7.16) we obtain

$$\mathcal{T}_{11}(p) = \sum_{d_1} \sum_{d_2} \frac{1}{d_1 d_2} \sum_k \frac{a_2(d_1 d_2 k) \mu(k)}{k \varphi(k)} \sum_{(l_2, k)=1} a_1(d_2 l_2) \sum_{(l_1, d_2 k)=1} \kappa(d_1 l_1) \Delta\left(\frac{l_1 l_2}{pk}\right). \quad (7.18)$$

The innermost sum is, by the Mellin transform, equal to

$$\frac{1}{2\pi i} \int_{(3/2)} \left(\sum_{(l, d_2 k)=1} \frac{\kappa(d_1 l)}{l^s} \right) \left(\frac{pk}{l_2} \right)^s \delta(s) ds \quad (7.19)$$

(here we have rewritten l for l_1). In view of (7.2), we can assume that

$$d_2 l_2 < PT^{-2}, \quad d_1 d_2 k < PT^{-2}.$$

Every l with $(l, d_2 k) = 1$ can be uniquely written as $l = hr$ such that $h \in \mathcal{N}(d_1)$, $(h, d_2 k) = 1$ and $(r, d_1 d_2 k) = 1$. Hence,

$$\sum_{(l, kd_2)=1} \frac{\kappa(d_1 l)}{l^s} = \tilde{\kappa}(d_1; d_2 k, s) \lambda(d_1 d_2 k, s) \frac{\zeta(s + \beta_1) \zeta(s + \beta_2) \zeta(s + \beta_3)}{\zeta(s)},$$

By the simple bounds

$$|\tilde{\kappa}(d_1; d_2 k, s)| \leq \tau_5(d_1) \prod_{q|d_1} \left| 1 + \frac{c}{q^\sigma} \right|, \quad |\lambda(m, s)| \leq \prod_{q|m} \left| 1 + \frac{c}{q^\sigma} \right|$$

for $\sigma > 9/10$, we can move the contour of integration in (7.19) to the vertical segments

$$\begin{aligned} s &= 1 + \alpha + it && \text{with } |t| \geq D, \\ s &= 1 - \mathcal{L}^{-1} + it && \text{with } |t| \leq D \end{aligned}$$

and to the two connecting horizontal segments

$$s = \sigma \pm iD \quad \text{with } 1 - \mathcal{L}^{-1} \leq \sigma \leq 1 + \alpha.$$

This yields, by Lemma 5.2 (i) and standard estimates, that the integral (7.18) is equal to the sum of the residues of the integrand at $s = 1 - \beta_j$, $1 \leq j \leq 3$, plus an acceptable error. Namely we have

$$\sum_{(l, kd_2)=1} \kappa(d_1 l) \Delta \left(\frac{ll_2}{pk} \right) = \sum_{1 \leq j \leq 3} \mathcal{R}_j \tilde{\kappa}_{0j}(d_1; d_2 k) \lambda_{0j}(d_1 d_2 k) \left(\frac{pk}{l_2} \right)^{1-\beta_j} + O \left(\frac{pk \varepsilon_1}{l_2} \right),$$

where \mathcal{R}_j is the residue of the function

$$\frac{\zeta(s + \beta_1) \zeta(s + \beta_2) \zeta(s + \beta_3)}{\zeta(s)} \delta(s)$$

at $s = 1 - \beta_j$, and where

$$\varepsilon_1 = \exp\{-c\mathcal{L}^{1/10}\}.$$

This yields

$$\begin{aligned} & \sum_{(l_2, k)=1} a_1(d_2 l_2) \sum_{(l_1, k d_2)=1} \kappa(d_1 l_1) \Delta\left(\frac{l_1 l_2}{pk}\right) \\ &= \sum_{1 \leq j \leq 3} \mathcal{R}_j(pk)^{1-\beta_j} \sum_{d=d_1 d_2} \tilde{\kappa}_{0j}(d_1; d_2 k) \lambda_{0j}(d_1 d_2 k) \sum_{(l, k)=1} \frac{a_1(d_2 l)}{l^{1-\beta_j}} + O(Pk\varepsilon_1). \end{aligned}$$

(here we have rewritten l for l_2). Inserting this into (7.18) and rearranging the terms we obtain

$$\mathcal{T}_{11}(p) = \sum_{1 \leq j \leq 3} \mathcal{R}_j p^{1-\beta_j} S_j^*(\mathbf{a}_1, \mathbf{a}_2) + O(P\varepsilon_1) \quad (7.20)$$

where

$$S_j^*(\mathbf{a}_1, \mathbf{a}_2) = \sum_d \sum_{d_1} \sum_k \frac{a_2(d_1 dk) \tilde{\kappa}_{0j}(d_1, dk) \lambda_{0j}(d_1 dk) \mu(k)}{d_1 d \varphi(k) k^{\beta_j}} \sum_{(l, k)=1} \frac{a_1(dl)}{l^{1-\beta_j}}.$$

By the Möbius inversion,

$$\begin{aligned} S_j^*(\mathbf{a}_1, \mathbf{a}_2) &= \sum_d \sum_{d_1} \sum_k \frac{a_2(d_1 dk) \tilde{\kappa}_{0j}(d_1; dk) \lambda_{0j}(d_1 dk) \mu(k)}{d_1 d \varphi(k) k^{\beta_j}} \\ &\quad \times \sum_l \frac{a_1(dl)}{l^{1-\beta_j}} \left(\sum_{r|(k, l)} \mu(r) \right). \end{aligned}$$

This yields, by substituting $k = rk_1$, $l = rm$, $n = d_1 k_1$ and changing the order of summation,

$$S_j^*(\mathbf{a}_1, \mathbf{a}_2) = \sum_r \sum_d \frac{|\mu(r)|}{dr \varphi(r)} \sum_m \frac{a_1(drm)}{m^{1-\beta_j}} \sum_n \frac{a_2(drn) \lambda_{0j}(drn)}{n} \sum_{\substack{n=d_1 k_1 \\ (k_1, r)=1}} \frac{\tilde{\kappa}_{0j}(d_1, drk_1) \mu(k_1) k_1^{1-\beta_j}}{\varphi(k_1)}.$$

Since

$$\lambda_{0j}(drn) = \lambda_{0j}(dr) \tilde{\lambda}_{0j}(n, dr),$$

it follows that

$$\lambda_{0j}(drn) \sum_{\substack{n=d_1 k_1 \\ (k_1, r)=1}} \frac{\tilde{\kappa}_{0j}(d_1, drk_1) \mu(k_1) k_1^{1-\beta_j}}{\varphi(k_1)} = \lambda_{0j}(dr) \xi_{0j}(n; d, r).$$

Hence

$$S_j^*(\mathbf{a}_1, \mathbf{a}_2) = S_j(\mathbf{a}_1, \mathbf{a}_2). \quad (7.21)$$

On the other hand, by Lemma 5.2 (ii) and direct calculation we have

$$i\mathcal{R}_1 p^{-\beta_1} = \frac{1}{2\alpha} + O(\mathcal{L}),$$

$$i\mathcal{R}_2 p^{-\beta_2} = \frac{2}{\alpha} + O(\mathcal{L}),$$

$$i\mathcal{R}_3 p^{-\beta_3} = \frac{3}{2\alpha} + O(\mathcal{L}).$$

Combining these with (7.20) and (7.21) we obtain (7.10), and complete the proof of Proposition 7.1. \square

8. Evaluation of Ξ_{11}

We first prove a general result as follows.

Lemma 8.1. *For any \mathbf{a}_1 and \mathbf{a}_2 satisfying (7.2),*

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) A(\mathbf{a}_1; \rho, \psi) A(\mathbf{a}_2, 1 - \rho, \bar{\psi}) \omega(\rho) = \Theta_1(\mathbf{a}_1, \mathbf{a}_2) + \overline{\Theta_1(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1)} + o(\mathcal{P}).$$

Proof. Write

$$\tilde{\mathcal{C}}(s, \psi) = -i \frac{M(s + \beta_1, \psi) M(s + \beta_2, \psi) M(s + \beta_3, \psi)}{M(s, \psi)}.$$

Assume $\psi \in \Psi_1$. By Proposition 2.2, we can choose a rectangle \mathfrak{R} with vertices at

$$s_0 \pm \alpha + i\mathcal{L}_1^\pm$$

such that $\mathcal{L}_1^\pm = \pm\mathcal{L}_1 + O(\alpha)$ and such that the set of zeros of $L(s, \psi)$ inside \mathfrak{R} is exactly $\tilde{Z}(\psi)$. Further, without loss of generality, we can assume that $|s - \rho| \gg \alpha$ if $s \in \mathfrak{R}$ and ρ is a zero of $L(s, \psi)$. By Lemma 5.9, the residue theorem and a simple bound for $\omega(s)$,

$$\begin{aligned} & \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) A(\mathbf{a}_1; \rho, \psi) A(\mathbf{a}_2, 1 - \rho, \bar{\psi}) \omega(\rho) \\ &= \frac{1}{2\pi i} \int_{\mathfrak{R}} \tilde{\mathcal{C}}(s, \psi) A(\mathbf{a}_1; s, \psi) A(\mathbf{a}_2, 1 - s, \bar{\psi}) \omega(s) ds \\ &= \tilde{I}_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) - \tilde{I}_1^-(\mathbf{a}_1, \mathbf{a}_2; \psi) + O(\varepsilon) \end{aligned} \tag{8.1}$$

where

$$\tilde{I}_1^\pm(\mathbf{a}_1, \mathbf{a}_2; \psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(\pm\alpha)} \tilde{\mathcal{C}}(s, \psi) A(\mathbf{a}_1; s, \psi) A(\mathbf{a}_2, 1 - s, \bar{\psi}) \omega(s) ds.$$

Write $s' = 1 - \bar{s}$. If

$$s = \alpha + s_0 + iv \in \mathcal{J}(\alpha).$$

then

$$s' = -\alpha + s_0 + iv \in \mathcal{J}(-\alpha).$$

By (2.11) and analytic continuation, for $s \in \mathcal{J}(\alpha)$ we have

$$\overline{\tilde{\mathcal{C}}(s, \psi)} = -\tilde{\mathcal{C}}(s', \psi)$$

and

$$\overline{A(\mathbf{a}_1; s, \psi)A(\mathbf{a}_2, 1 - s, \bar{\psi})\omega(s)} = A(\bar{\mathbf{a}}_2; s', \psi)A(\bar{\mathbf{a}}_1, 1 - s', \bar{\psi})\omega(s').$$

These together imply that

$$-\tilde{I}_1^-(\mathbf{a}_1, \mathbf{a}_2; \psi) = \overline{\tilde{I}_1^+(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1; \psi)}.$$

Hence, by (8.6),

$$\sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi)A(\mathbf{a}_1; \rho, \psi)A(\mathbf{a}_2, 1 - \rho, \bar{\psi})\omega(\rho) = \tilde{I}_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) + \overline{\tilde{I}_1^+(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1; \psi)} + O(\varepsilon).$$

Write

$$I_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(\alpha)} \mathcal{C}(s, \psi)A(\mathbf{a}_1; s, \psi)A(\mathbf{a}_2, 1 - s, \bar{\psi})\omega(s) ds.$$

By Lemma 5.2 and 5.9,

$$\tilde{I}_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) - I_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) \ll \mathcal{L}^{-114} \int_{\mathcal{J}(\alpha)} \left| L(s + \beta_2, \psi)L(s + \beta_3, \psi)A(\mathbf{a}_1; s, \psi)A(\mathbf{a}_2, 1 - s, \bar{\psi})\omega(s) ds \right|.$$

By Cauchy's inequality, Lemma 6.1, and the second assertion of Lemma 3.3, for $s \in \mathcal{J}(\alpha)$,

$$\sum_{\psi \in \Psi_1} |L(s + \beta_2, \psi)L(s + \beta_3, \psi)|^2 \ll P^2 \mathcal{L}^{36},$$

$$\sum_{\psi \in \Psi_1} |A(\mathbf{a}_1; s, \psi)A(\mathbf{a}_2, 1 - s, \bar{\psi})|^2 \ll P^2 \mathcal{L}^{36}.$$

These estimates together with (7.4) and (2.9) imply

$$\sum_{\psi \in \Psi_1} (\tilde{I}_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) - I_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi)) = o(\mathcal{P}).$$

On the other hand, moving the segment $\mathcal{J}(\alpha)$ to $\mathcal{J}(1)$ gives

$$\sum_{\psi \in \Psi_1} I_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) = \Theta_1(\mathbf{a}_1, \mathbf{a}_2) + O(\varepsilon).$$

Hence

$$\sum_{\psi \in \Psi_1} \tilde{I}_1^+(\mathbf{a}_1, \mathbf{a}_2; \psi) = \Theta_1(\mathbf{a}_1, \mathbf{a}_2) + o(\mathcal{P}).$$

For the sum of $\overline{\tilde{I}_1^+(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1; \psi)}$ a similar result holds. This completes the proof. \square

Recall that $H_1(s, \psi)$ and $H_2(s, \psi)$ are given by (2.27). Write

$$\Xi_1 = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |H_1(\rho, \psi) + Z(\rho, \chi\psi) \overline{H_2(s, \psi)}|^2 \omega(\rho).$$

Since $|Z(s, \chi\psi)| = 1$ if $\sigma = 1/2$, it follows that

$$\Xi_1 = \Xi_{11} + \Xi_{12} + 2\Re\{\Xi_{13}\} \quad (8.2)$$

where

$$\Xi_{11} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |H_1(\rho, \psi)|^2 \omega(\rho), \quad (8.3)$$

$$\Xi_{12} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |H_2(\rho, \psi)|^2 \omega(\rho), \quad (8.4)$$

$$\Xi_{13} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) Z(\rho, \chi\psi)^{-1} H_1(\rho, \psi) H_2(\rho, \psi) \omega(\rho). \quad (8.5)$$

We may write

$$H_{1j}(s, \psi) = \sum_n \frac{\varkappa_j(n) \chi\psi(n)}{n^s} \quad (8.6)$$

where

$$\varkappa_1(n) = \begin{cases} (1 - \log n / \log P_1)(P_1/n)^{\beta_6} & \text{if } n < P_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varkappa_2(n) = \begin{cases} (1 - \log n / \log P_2)(P_2/n)^{\beta_7} & \text{if } n < P_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varkappa_3(n) = \begin{cases} (1 - \log n / \log P_3)(P_3/n)^{\beta_6} & \text{if } n < P_3, \\ 0 & \text{otherwise.} \end{cases}$$

In this section we evaluate Ξ_{11} . By Lemma 8.1,

$$\Xi_{11} = 2\Re\{\Theta_1(\mathbf{a}_{11}, \mathbf{a}_{21})\} + o(\mathcal{P}) \quad (8.7)$$

with

$$a_{11}(n) = \chi(n)(\varkappa_1(n) + \iota_2 \varkappa_2(n)), \quad a_{21}(n) = \overline{a_{11}(n)}. \quad (8.8)$$

By Proposition 7.1, our goal is reduced to evaluating the sum

$$S_j(\mathbf{a}_{11}, \mathbf{a}_{21}) = \sum_d \sum_r \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) (\varkappa_1(drm) + \iota_2 \varkappa_2(drm))}{m^{1-\beta_j}} \right) \\ \times \left(\sum_n \frac{\chi(n) (\bar{\varkappa}_1(drn) + \bar{\iota}_2 \bar{\varkappa}_2(drn)) \xi_{0j}(n; d, r)}{n} \right).$$

Write

$$\beta_4 = \beta_1, \quad \beta_5 = \beta_2,$$

so that

$$\{\beta_j, \beta_{j+1}, \beta_{j+2}\} = \{\beta_1, \beta_2, \beta_3\}.$$

Lemma 8.2. *Suppose $T < x < P$. Then for $\mu = 6, 7$,*

$$\sum_{m < x} \frac{\chi(m)}{m^{1-\beta_j}} \left(\frac{x}{m} \right)^{\beta_\mu} \log \frac{x}{m} = L'(1, \chi) \mathcal{F}_{j\mu}(x) + O(\mathcal{L}^{-6})$$

where

$$\mathcal{F}_{j\mu}(x) = (1 + (\beta_\mu - \beta_j) \log x) x^{\beta_\mu}.$$

Proof. The sum is equal to

$$\frac{1}{2\pi i} \int_{(1)} L(1 - \beta_j + s, \chi) \frac{x^s ds}{(s - \beta_\mu)^2}.$$

We move the contour of integration to the vertical segments

$$s = \alpha + it \quad \text{with } |t| \geq D, \\ s = -\mathcal{L}^{-1} + it \quad \text{with } |t| \leq D$$

and to the two connecting horizontal segments

$$s = \sigma + \pm iD, \quad \text{with } -\mathcal{L}^{-1} \leq \sigma \leq \alpha.$$

It follows by Lemma 5.6 that

$$\sum_{m < x} \frac{\chi(m)}{m^{1-\beta_j}} \left(\frac{x}{m} \right)^{\beta_\mu} \log \frac{x}{m} = \frac{1}{2\pi i} \int_{|s|=5\alpha} L(1 - \beta_j + s, \chi) \frac{x^s ds}{(s - \beta_\mu)^2} + O(\varepsilon_1) \\ = L'(1, \chi) \cdot \frac{1}{2\pi i} \int_{|s|=5\alpha} \frac{x^s (s - \beta_j)}{(s - \beta_\mu)^2} ds + O(\mathcal{L}^{-6}).$$

The result now follows by direct calculation. \square

The sum involving $\xi_{0j}(n; d, r)$ is more involved. We need the following lemma that will be proved in Appendix A.

Lemma 8.3. *Suppose $dr < PT^{-2}$. The function*

$$\mathcal{U}_j(d, r; s) := \frac{L(s, \chi)}{L(s + \beta_{j+1}, \chi)L(s + \beta_{j+2}, \chi)} \sum_n \frac{\chi(n)\xi_{0j}(n; d, r)}{n^s}$$

is analytic and it satisfies

$$|\mathcal{U}_j(d, r; s)| < c \prod_{q|dr} (1 + cq^{-\sigma})$$

for $\sigma > 9/10$. Further, if $|s - 1| \leq 5\alpha$, then

$$\mathcal{U}_j(d, r; s) = \Pi(d, r) + O(\mathcal{L}^{-8})$$

where

$$\Pi(d, r) = \prod_{q|dr} \frac{1}{1 - \chi(q)q^{-1}} \prod_{\substack{(q,r)=1 \\ q|d}} \frac{1 - q^{-1} - \chi(q)q^{-1}}{1 - q^{-1}}.$$

Lemma 8.4. *Suppose $dr < PT^{-2}$ and $T < x < P$. Then for $\mu = 6, 7$,*

$$\sum_{n < x} \frac{\chi(n)\xi_{0j}(n; d, r)}{n} \left(\frac{x}{n}\right)^{-\beta_\mu} \log \frac{x}{n} = L'(1, \chi)\Pi(d, r)\mathcal{G}_{j\mu}(x) + O(\mathcal{L}^{-6})$$

where

$$\mathcal{G}_{j\mu}(x) = \frac{\beta_{j+1}\beta_{j+2}}{\beta_\mu^2} + \left(1 - \frac{\beta_{j+1}\beta_{j+2}}{\beta_\mu^2} - \frac{(\beta_{j+1} - \beta_\mu)(\beta_{j+2} - \beta_\mu)}{\beta_\mu} \log x\right) x^{-\beta_\mu}.$$

Proof. The sum is equal to

$$\frac{1}{2\pi i} \int_{(1)} \left(\sum_n \frac{\chi(n)\xi_{0j}(n; d, r)}{n^{1+s}} \right) \frac{x^s ds}{(s + \beta_\mu)^2}. \quad (8.9)$$

The contour of integration is moved in the same way as in the proof of Lemma 8.2. Recall that

$$\sum_n \frac{\chi(n)\xi_{0j}(n; d, r)}{n^{1+s}} = \frac{L(1 + s + \beta_{j+1}, \chi)L(1 + s + \beta_{j+2}, \chi)}{L(1 + s, \chi)} \mathcal{U}_j(d, r, 1 + s).$$

By Lemma 5.5 and 5.6, for $|s| = 5\alpha$ we have

$$\frac{L(1 + s + \beta_{j+1}, \chi)L(1 + s + \beta_{j+2}, \chi)}{L(1 + s, \chi)} = L'(1, \chi) \frac{(s + \beta_{j+1})(s + \beta_{j+2})}{s} + O(\mathcal{L}^{-15}).$$

Combining these results with Lemma 8.3, we find that the integral (8.9) is equal to

$$L'(1, \chi)\Pi(d, r) \cdot \frac{1}{2\pi i} \int_{|s|=5\alpha} \frac{(s + \beta_{j+1})(s + \beta_{j+2})}{s} \frac{x^s ds}{(s + \beta_\mu)^2} + O(\mathcal{L}^{-6}).$$

The result now follows by direct calculation. \square

By Lemma 8.2 with $x = P_1/dr$ and $x = P_2/dr$ respectively we have

$$\sum_m \frac{\chi(m)\mathfrak{z}_1(drm)}{m^{1-\beta_j}} = \frac{L'(1, \chi)}{\log P_1} \mathcal{F}_{j6}(P_1/dr) + O(\mathcal{L}^{-15})$$

if $dr < P_1/T$, and

$$\sum_m \frac{\chi(m)\mathfrak{z}_2(drm)}{m^{1-\beta_j}} = \frac{L'(1, \chi)}{\log P_2} \mathcal{F}_{j7}(P_2/dr) + O(\mathcal{L}^{-15})$$

if $dr < P_2/T$. On the other hand, by Lemma 8.3 with $x = P_1/dr$ and $x = P_2/dr$ respectively we have

$$\sum_n \frac{\chi(n)\bar{\mathfrak{z}}_1(drn)\xi_{0j}(n; d, r)}{n} = \frac{L'(1, \chi)\Pi(d, r)}{\log P_1} \mathcal{G}_{j6}(P_1/dr) + O(\mathcal{L}^{-15})$$

if $dr < P_1/T$, and

$$\sum_n \frac{\chi(n)\bar{\mathfrak{z}}_2(drn)\xi_{0j}(n; d, r)}{n} = \frac{L'(1, \chi)\Pi(d, r)}{\log P_2} \mathcal{G}_{j7}(P_2/dr) + O(\mathcal{L}^{-15})$$

if $dr < P_2/T$. Gathering these results together we conclude, by simple approximation, that

$$\begin{aligned} S_j(\mathbf{a}_{11}, \mathbf{a}_{21}) &= L'(1, \chi)^2 \sum_{dr < P_2} \frac{|\mu(r)\chi(dr)|}{dr\varphi(r)} \lambda_{0j}(dr)\Pi(d, r) \\ &\quad \times \left(\frac{\mathcal{F}_{j6}(P_1/dr)}{\log P_1} + \iota_2 \frac{\mathcal{F}_{j7}(P_2/dr)}{\log P_2} \right) \left(\frac{\mathcal{G}_{j6}(P_1/dr)}{\log P_1} + \bar{\iota}_2 \frac{\mathcal{G}_{j7}(P_2/dr)}{\log P_2} \right) \\ &\quad + L'(1, \chi)^2 \sum_{P_2 \leq dr < P_1} \frac{|\mu(r)\chi(dr)|}{dr\varphi(r)} \lambda_{0j}(dr)\Pi(d, r) \frac{\mathcal{F}_{j6}(P_1/dr)\mathcal{G}_{j6}(P_1/dr)}{(\log P_1)^2} + o(\alpha). \end{aligned}$$

It can be shown, by verifying the case $n = q^k$, that

$$\sum_{n=dr} \frac{|\mu(r)|}{r} \prod_{\substack{(q,r)=1 \\ q|d}} (1 - q^{-1} - \chi(q)q^{-1}) = \prod_{q|n} (1 - \chi(q)q^{-1}),$$

so that

$$\sum_{n=dr} \frac{|\mu(r)|}{\varphi(r)} \Pi(d, r) = \frac{n}{\varphi(n)}. \quad (8.10)$$

It follows, by substituting $n = dr$, that

$$\begin{aligned} S_j(\mathbf{a}_{11}, \mathbf{a}_{21}) &= L'(1, \chi)^2 \sum_{n < P_2} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \\ &\quad \times \left(\frac{\mathcal{F}_{j6}(P_1/n)}{\log P_1} + \iota_2 \frac{\mathcal{F}_{j7}(P_2/n)}{\log P_2} \right) \left(\frac{\mathcal{G}_{j6}(P_1/n)}{\log P_1} + \bar{\iota}_2 \frac{\mathcal{G}_{j7}(P_2/n)}{\log P_2} \right) \\ &+ L'(1, \chi)^2 \sum_{P_2 \leq n < P_1} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \frac{\mathcal{F}_{j6}(P_1/n) \mathcal{G}_{j6}(P_1/n)}{(\log P_1)^2} + o(\alpha). \end{aligned}$$

Since for $n < P$,

$$\lambda_{0j}(n) = \prod_{q|n} (1 - q^{-1})^2 (1 + O(\alpha \log q/q)) = \frac{\varphi(n)^2}{n^2} + O(\alpha_1)$$

and, for $x < P$,

$$\sum_{n < x} \frac{|\chi(n)| \varphi(n)}{n^2} = \frac{\varphi(D)}{D} \left(\prod_{(q,D)=1} (1 - q^{-2}) \right) \log x + O(\log \mathcal{L}) = \frac{6}{\pi^2} \left(\prod_{q|D} \frac{q}{q+1} \right) \log x + O(\log \mathcal{L}),$$

(see [T, 1.2.12]), it follows by partial integration that

$$\begin{aligned} S_j(\mathbf{a}_{11}, \mathbf{a}_{21}) &= \mathbf{a} \int_1^{P_2} \left(\frac{\mathcal{F}_{j6}(P_1/x)}{\log P_1} + \iota_2 \frac{\mathcal{F}_{j7}(P_2/x)}{\log P_2} \right) \left(\frac{\mathcal{G}_{j6}(P_1/x)}{\log P_1} + \bar{\iota}_2 \frac{\mathcal{G}_{j7}(P_2/x)}{\log P_2} \right) \frac{dx}{x} \\ &+ \frac{\mathbf{a}}{(\log P_1)^2} \int_{P_2}^{P_1} \frac{\mathcal{F}_{j6}(P_1/x) \mathcal{G}_{j6}(P_1/x)}{x} dx + o(\alpha). \end{aligned} \quad (8.11)$$

This yields, by the change of variable $x \rightarrow P_1/x$ or $x \rightarrow P_2/x$,

$$\begin{aligned} S_j(\mathbf{a}_{11}, \mathbf{a}_{21}) &= \frac{\mathbf{a}}{(\log P_1)^2} \int_1^{P_1} \frac{\mathcal{F}_{j6}(x) \mathcal{G}_{j6}(x)}{x} dx + \frac{\mathbf{a} |\iota_2|^2}{(\log P_2)^2} \int_1^{P_2} \frac{\mathcal{F}_{j7}(x) \mathcal{G}_{j7}(x)}{x} dx \\ &+ \frac{\mathbf{a} \iota_2}{(\log P_1)(\log P_2)} \int_1^{P_2} \frac{\mathcal{F}_{j7}(x) \mathcal{G}_{j6}(P^{0.004} T^{10} x)}{x} dx \\ &+ \frac{\mathbf{a} \bar{\iota}_2}{(\log P_1)(\log P_2)} \int_1^{P_2} \frac{\mathcal{F}_{j6}(P^{0.004} T^{10} x) \mathcal{G}_{j7}(x)}{x} dx + o(\alpha), \end{aligned} \quad (8.12)$$

since $P_1/P_2 = P^{0.004} T^{10}$. By direct calculation, for $0 \leq z \leq 1$ we have

$$\mathcal{F}_{j6}(P^z) = \mathfrak{f}_{j6}(z) + O(\mathcal{L}^{-8}), \quad \mathcal{G}_{j6}(P^z) = \mathfrak{g}_{j6}(z) + O(\mathcal{L}^{-8}),$$

$$\mathcal{F}_{j7}(P^z) = \mathfrak{f}_{j7}(z) + O(\mathcal{L}^{-8}), \quad \mathcal{G}_{j7}(P^z) = \mathfrak{g}_{j7}(z) + O(\mathcal{L}^{-8})$$

where

$$\mathfrak{f}_{16}(z) = \left(1 + \frac{\pi i}{2}z\right)e^{(3\pi i/2)z}, \quad \mathfrak{g}_{16}(z) = \frac{8}{3} + \left(-\frac{5}{3} - \frac{\pi i}{2}z\right)e^{(-3\pi i/2)z}, \quad (8.13)$$

$$\mathfrak{f}_{26}(z) = \left(1 - \frac{\pi i}{2}z\right)e^{(3\pi i/2)z}, \quad \mathfrak{g}_{26}(z) = \frac{4}{3} + \left(-\frac{1}{3} + \frac{\pi i}{2}z\right)e^{(-3\pi i/2)z}, \quad (8.14)$$

$$\mathfrak{f}_{36}(z) = \left(1 - \frac{3\pi i}{2}z\right)e^{(3\pi i/2)z}, \quad \mathfrak{g}_{36}(z) = \frac{8}{9} + \left(\frac{1}{9} + \frac{\pi i}{6}z\right)e^{(-3\pi i/2)z}. \quad (8.15)$$

$$\mathfrak{f}_{17}(z) = \left(1 + \frac{3\pi i}{2}z\right)e^{(5\pi i/2)z}, \quad \mathfrak{g}_{17}(z) = \frac{24}{25} + \left(\frac{1}{25} + \frac{\pi i}{10}z\right)e^{(-5\pi i/2)z}, \quad (8.16)$$

$$\mathfrak{f}_{27}(z) = \left(1 + \frac{\pi i}{2}z\right)e^{(5\pi i/2)z}, \quad \mathfrak{g}_{27}(z) = \frac{12}{25} + \left(\frac{13}{25} + \frac{3\pi i}{10}z\right)e^{(-5\pi i/2)z}, \quad (8.17)$$

$$\mathfrak{f}_{37}(z) = \left(1 - \frac{\pi i}{2}z\right)e^{(5\pi i/2)z}, \quad \mathfrak{g}_{37}(z) = \frac{8}{25} + \left(\frac{17}{25} - \frac{3\pi i}{10}z\right)e^{(-5\pi i/2)z}. \quad (8.18)$$

Substituting $x = P^z$ we obtain

$$\begin{aligned} \frac{1}{(\log P_1)^2} \int_1^{P_1} \frac{\mathcal{F}_{j6}(x)\mathcal{G}_{j6}(x)}{x} dx &= \frac{1}{(0.504)^2 \log P} \int_0^{0.504} \mathfrak{f}_{j6}(z)\mathfrak{g}_{j6}(z) dz + o(\alpha), \\ \frac{1}{(\log P_2)^2} \int_1^{P_2} \frac{\mathcal{F}_{j7}(x)\mathcal{G}_{j7}(x)}{x} dx &= \frac{1}{(0.5)^2 \log P} \int_0^{0.5} \mathfrak{f}_{j7}(z)\mathfrak{g}_{j7}(z) dz + o(\alpha), \\ \frac{1}{(\log P_1)(\log P_2)} \int_1^{P_2} \frac{\mathcal{F}_{j7}(x)\mathcal{G}_{j6}(P^{0.004}T^{10}x)}{x} dx \\ &= \frac{1}{(0.5)(0.504) \log P} \int_0^{0.5} \mathfrak{f}_{j7}(z)\mathfrak{g}_{j6}(z + 0.004) dz + o(\alpha), \\ \frac{1}{(\log P_1)(\log P_2)} \int_1^{P_2} \frac{\mathcal{F}_{j6}(P^{0.004}T^{10}x)\mathcal{G}_{j7}(x)}{x} dx \\ &= \frac{1}{(0.5)(0.504) \log P} \int_0^{0.5} \mathfrak{f}_{j6}(z + 0.004)\mathfrak{g}_{j7}(z) dz + o(\alpha). \end{aligned}$$

Inserting these into (8.13) we obtain

$$\frac{1}{2\alpha}S_1(\mathbf{a}_{11}, \mathbf{a}_{21}) + \frac{2}{\alpha}S_2(\mathbf{a}_{11}, \mathbf{a}_{21}) + \frac{3}{2\alpha}S_3(\mathbf{a}_{11}, \mathbf{a}_{21}) = \mathbf{a}(b_{11} + \iota_2 b_{21} + \bar{\iota}_2 b_{12} + |\iota_2|^2 b_{22}) + o(1)$$

where

$$b_{11} = \frac{1}{0.504^2 \pi} \int_0^{0.504} \left(\frac{1}{2} \mathfrak{f}_{16}(z)\mathfrak{g}_{16}(z) + 2\mathfrak{f}_{26}(z)\mathfrak{g}_{26}(z) + \frac{3}{2}\mathfrak{f}_{36}(z)\mathfrak{g}_{36}(z) \right) dz, \quad (8.19)$$

$$b_{22} = \frac{1}{0.5^2\pi} \int_0^{0.5} \left(\frac{1}{2} \mathbf{f}_{17}(z) \mathbf{g}_{17}(z) + 2 \mathbf{f}_{27}(z) \mathbf{g}_{27}(z) + \frac{3}{2} \mathbf{f}_{37}(z) \mathbf{g}_{37}(z) \right) dz, \quad (8.20)$$

$$b_{21} = \frac{1}{(0.504)(0.5)\pi} \times \int_0^{0.5} \left(\frac{1}{2} \mathbf{f}_{17}(z) \mathbf{g}_{16}(z + 0.004) + 2 \mathbf{f}_{27}(z) \mathbf{g}_{26}(z + 0.004) + \frac{3}{2} \mathbf{f}_{37}(z) \mathbf{g}_{36}(z + 0.004) \right) dz. \quad (8.21)$$

$$b_{12} = \frac{1}{(0.504)(0.5)\pi} \times \int_0^{0.5} \left(\frac{1}{2} \mathbf{f}_{16}(z + 0.004) \mathbf{g}_{17}(z) + 2 \mathbf{f}_{26}(z + 0.004) \mathbf{g}_{27}(z) + \frac{3}{2} \mathbf{f}_{36}(z + 0.004) \mathbf{g}_{37}(z) \right) dz. \quad (8.22)$$

It follows by Proposition 7.1 that

$$\Theta_1(\mathbf{a}_{11}, \mathbf{a}_{21}) = (b_{11} + \iota_2 b_{21} + \bar{\iota}_2 b_{12} + |\iota_2|^2 b_{22}) \mathbf{aP} + o(\mathcal{P}).$$

This yields, by (8.7),

$$\Xi_{11} = \mathbf{c}_1 \mathbf{aP} + o(\mathcal{P}) \quad (8.23)$$

where

$$\mathbf{c}_1 = c_{11} + \iota_2 c_{21} + \bar{\iota}_2 c_{12} + |\iota_2|^2 c_{22}$$

with

$$c_{11} = b_{11} + \bar{b}_{11},$$

$$c_{22} = b_{22} + \bar{b}_{22},$$

$$c_{12} = b_{12} + \bar{b}_{21},$$

$$c_{21} = \bar{c}_{12}.$$

Let ϵ be a complex number satisfying $|\epsilon| < 10^{-5}$, not necessarily the same in each occurrence. Numerical calculation shows that

$$c_{11} = 3.61226 + \epsilon/2,$$

$$c_{22} = 1.32215 + \epsilon/2,$$

$$c_{12} = -0.45757 - 0.18179i + \epsilon/\sqrt{2}.$$

It follows that

$$\mathbf{c}_1 < 6.9955. \quad (8.24)$$

9. Evaluation of Ξ_{12}

In this section we evaluate Ξ_{12} . By Lemma 8.1,

$$\Xi_{12} = 2\Re\{\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{22})\} + o(\mathcal{P}) \quad (9.1)$$

with with

$$a_{12}(n) = \chi(n)(\bar{\iota}_3 \varkappa_3(n) + \bar{\iota}_4 \varkappa_2(n)), \quad a_{22}(n) = \overline{a_{21}(n)}. \quad (9.2)$$

Hence

$$\begin{aligned} S_j(\mathbf{a}_{12}, \mathbf{a}_{22}) &= \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m)(\bar{\iota}_3 \varkappa_3(drm) + \bar{\iota}_4 \varkappa_2(drm))}{m^{1-\beta_j}} \right) \\ &\quad \times \left(\sum_n \frac{\chi(n)(\iota_3 \bar{\varkappa}_3(drn) + \iota_4 \bar{\varkappa}_2(drn)) \xi_{0j}(n; d, r)}{n} \right). \end{aligned}$$

By Lemma 8.2 and 8.4, for $dr < P_3/T$,

$$\sum_m \frac{\chi(m) \varkappa_3(drm)}{m^{1-\beta_j}} = \frac{L'(1, \chi)}{\log P_3} \mathcal{F}_{j6}(P_3/dr) + O(\mathcal{L}^{-6})$$

and

$$\sum_n \frac{\chi(n) \bar{\varkappa}_3(drn) \xi_{0j}(n; d, r)}{n} = \frac{L'(1, \chi) \Pi(d, r)}{\log P_3} \mathcal{G}_{j6}(P_3/dr) + O(\mathcal{L}^{-6}).$$

Thus, in a way similar to the proof of (8.12), we deduce that

$$\begin{aligned} S_j(\mathbf{a}_{12}, \mathbf{a}_{22}) &= \mathbf{a} \sum_{r < D} \int_1^{P_3} \left(\bar{\iota}_3 \frac{\mathcal{F}_{j6}(P_3/x)}{\log P_3} + \bar{\iota}_4 \frac{\mathcal{F}_{j7}(P_2/x)}{\log P_2} \right) \left(\iota_3 \frac{\mathcal{G}_{j6}(P_3/x)}{\log P_3} + \iota_4 \frac{\mathcal{G}_{j7}(P_2/x)}{\log P_2} \right) \frac{dx}{x} \\ &\quad + \frac{|\iota_4|^2 \mathbf{a}}{(\log P_2)^2} \int_{P_3}^{P_2} \mathcal{F}_{j7}(P_2/x) \mathcal{G}_{j7}(P_2/x) \frac{dx}{x} + o(\alpha). \end{aligned}$$

It follows, by the change of variables $x \rightarrow P_2/x$ or $x \rightarrow P_3/x$, that

$$\begin{aligned} S_j(\mathbf{a}_{12}, \mathbf{a}_{22}) &= \frac{\mathbf{a} |\iota_3|^2}{(\log P_3)^2} \int_1^{P_3} \frac{\mathcal{F}_{j6}(x) \mathcal{G}_{j6}(x)}{x} dx + \frac{\mathbf{a} |\iota_4|^2}{(\log P_2)^2} \int_1^{P_2} \frac{\mathcal{F}_{j7}(x) \mathcal{G}_{j7}(x)}{x} dx \\ &\quad + \frac{\mathbf{a} \bar{\iota}_3 \iota_4}{(\log P_2)(\log P_3)} \int_1^{P_3} \frac{\mathcal{F}_{j6}(x) \mathcal{G}_{j7}(P^{0.002} T^{-10} x)}{x} dx \\ &\quad + \frac{\mathbf{a} \bar{\iota}_4 \iota_3}{(\log P_2)(\log P_3)} \int_1^{P_3} \frac{\mathcal{F}_{j7}(P^{0.002} T^{-10} x) \mathcal{G}_{j6}(x)}{x} dx + o(\alpha), \end{aligned}$$

since $P_2/P_3 = P^{0.002} T^{-10}$. This yields, by substituting $x = P^z$,

$$\frac{1}{2\alpha} S_1(\mathbf{a}_{12}, \mathbf{a}_{22}) + \frac{2}{\alpha} S_2(\mathbf{a}_{12}, \mathbf{a}_{22}) + \frac{3}{2\alpha} S_3(\mathbf{a}_{12}, \mathbf{a}_{22}) = \mathbf{a} (|\iota_3|^2 b_{33} + \iota_3 \bar{\iota}_4 b_{34} + \iota_4 \bar{\iota}_3 b_{43} + |\iota_4|^2 b_{44}) + o(1)$$

where

$$b_{33} = \frac{1}{0.498^2\pi} \int_0^{0.498} \left(\frac{1}{2} \mathbf{f}_{16}(z) \mathbf{g}_{16}(z) + 2 \mathbf{f}_{26}(z) \mathbf{g}_{26}(z) + \frac{3}{2} \mathbf{f}_{36}(z) \mathbf{g}_{36}(z) \right) dz, \quad (9.3)$$

$$b_{44} = b_{22}, \quad (9.4)$$

$$b_{34} = \frac{1}{(0.504)(0.498)\pi} \int_0^{0.498} \left(\frac{1}{2} \mathbf{f}_{17}(z+0.002) \mathbf{g}_{16}(z) + 2 \mathbf{f}_{27}(z+0.002) \mathbf{g}_{26}(z) + \frac{3}{2} \mathbf{f}_{37}(z+0.002) \mathbf{g}_{36}(z) \right) dz, \quad (9.5)$$

$$b_{43} = \frac{1}{(0.504)(0.498)\pi} \int_0^{0.498} \left(\frac{1}{2} \mathbf{f}_{16}(z) \mathbf{g}_{17}(z+0.002) + 2 \mathbf{f}_{26}(z) \mathbf{g}_{27}(z+0.002) + \frac{3}{2} \mathbf{f}_{36}(z) \mathbf{g}_{37}(z+0.002) \right) dz. \quad (9.6)$$

It follows by Proposition 7.1 that

$$\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{22}) = (|\iota_3|^2 b_{33} + \iota_3 \bar{\iota}_4 b_{34} + \iota_4 \bar{\iota}_3 b_{43} + |\iota_4|^2 b_{44}) \mathbf{aP} + o(\mathcal{P})$$

This yields

$$\Xi_{12} = \mathbf{c}_2 \mathbf{aP} + o(\mathcal{P}) \quad (9.7)$$

where

$$\mathbf{c}_2 = |\iota_3|^2 c_{33} + \iota_3 \bar{\iota}_4 c_{34} + \iota_4 \bar{\iota}_3 c_{43} + |\iota_4|^2 c_{44}$$

with

$$c_{33} = b_{33} + \overline{b_{33}},$$

$$c_{44} = c_{22},$$

$$c_{34} = b_{34} + \overline{b_{43}},$$

$$c_{43} = \overline{c_{34}}.$$

Numerical calculation shows that

$$c_{33} = 3.69507 + \epsilon/2,$$

$$c_{34} = -0.4526 + 0.19474i + \epsilon/\sqrt{2}.$$

It follows that

$$\mathbf{c}_2 < 6.9955. \quad (9.8)$$

10. Proof of Proposition 2.4

The goal of this section is to prove Proposition 2.4. We continue to assume $1 \leq j \leq 3$.

Note that $\tilde{f}(z) \in \mathbf{R}$. If $\sigma = 1/2$, then

$$\overline{J_1(s, \psi)} = \sum_n \frac{\chi \bar{\psi}(n)}{n^{1-s}} \tilde{f}\left(\frac{\log n}{\log P}\right) = J_1(1-s, \bar{\psi}),$$

$$\overline{J_2(s, \psi)} = \sum_n \frac{\chi \bar{\psi}(n)}{n^{1-s}} \tilde{f}\left(\frac{\log n}{\log P} + 0.004 - \tilde{\alpha}\right) = J_2(1-s, \bar{\psi}).$$

By Lemma 8.1 we have

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) H_1(\rho, \psi) \overline{J_1(\rho, \psi)} \omega(\rho) = \Theta_1(\mathbf{a}_{11}, \mathbf{a}_{13}) + \overline{\Theta_1(\mathbf{a}_{13}, \mathbf{a}_{21})} + o(\mathcal{P})$$

with $a_{11}(n)$ and $a_{21}(n)$ given by (8.8), and with

$$a_{13}(n) = \chi(n) \tilde{f}\left(\frac{\log n}{\log P}\right),$$

and

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{C}^*(\rho, \psi) H_2(\rho, \psi) \overline{J_2(\rho, \psi)} \omega(\rho) = \Theta_1(\mathbf{a}_{14}, \mathbf{a}_{22}) + \overline{\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{14})} + o(\mathcal{P})$$

with $a_{12}(n)$ and $a_{22}(n)$ given by (9.2), and with

$$a_{14}(n) = \chi(n) \tilde{f}\left(\frac{\log n}{\log P} + 0.004 - \tilde{\alpha}\right),$$

Hence

$$\Xi_1^* = \Theta_1(\mathbf{a}_{11}, \mathbf{a}_{13}) + \overline{\Theta_1(\mathbf{a}_{13}, \mathbf{a}_{21})} + \Theta_1(\mathbf{a}_{14}, \mathbf{a}_{22}) + \overline{\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{14})} + o(\mathcal{P}). \quad (10.1)$$

Lemma 10.1. *Write*

$$\mathcal{V}_{1j}(y) = \sum_m \frac{\chi(m) \tilde{f}(\log(ym)/\log P)}{m^{1-\beta_j}}.$$

If $1 \leq y \leq P^{0.5}/T$, then

$$\mathcal{V}_{1j}(y) \ll T^{-c}; \quad (10.2)$$

if $P^{0.5} < y \leq P^{0.502}/T$, then

$$\mathcal{V}_{1j}(y) = \frac{500L'(1, \chi)}{\log P} (-1 - \beta_j \log(y/P^{0.5})) + O(\mathcal{L}^{-15}); \quad (10.3)$$

if $P^{0.502} < y \leq P^{0.504}/T$, then

$$\mathcal{V}_{1j}(y) = \frac{500L'(1, \chi)}{\log P} (1 - \beta_j \log(P^{0.504}/y)) + O(\mathcal{L}^{-15}); \quad (10.4)$$

if

$$y \in (P^{0.5}/T, P^{0.5}] \cup (P^{0.502}/T, P^{0.502}] \cup (P^{0.504}/T, P^{0.504}),$$

then

$$\mathcal{V}_{1j}(y) \ll \mathcal{L}^{-7}. \quad (10.5)$$

Proof. The inequalities (10.2) follow by the Polya-Vinogradov inequality and partial summation. Write

$$P'_1 = P^{0.504}, \quad P'_2 = P^{0.502}, \quad P'_3 = P^{0.5}.$$

Then

$$\tilde{f}(\log y / \log P) = \frac{500}{\log P} \cdot \frac{1}{2\pi i} \int_{(1)} \frac{(P'_1)^s - 2(P'_2)^s + (P'_3)^s}{y^s} \frac{ds}{s^2}. \quad (10.6)$$

In the case $P^{0.5} < y \leq P^{0.502}/T$,

$$\mathcal{V}_{1j}(y) = \frac{500}{\log P} \cdot \frac{1}{2\pi i} \int_{(1)} L(1 - \beta_j + s, \chi) \frac{(P'_1)^s - 2(P'_2)^s}{y^s} \frac{ds}{s^2}. \quad (10.7)$$

by (10.6). The contour of integration is moved in the same way as in the proof of lemma 8.1. Hence, by Lemma 5.8, the right side above is equal to

$$\frac{500L'(1, \chi)}{\log P} \cdot \frac{1}{2\pi i} \int_{|s|=5\alpha} \frac{(P'_1)^s - 2(P'_2)^s}{y^s} \frac{(s - \beta_j)ds}{s^2} + o(\mathcal{L}^{-15}).$$

This yields (11.3) since the residue of the function

$$\frac{(P'_1)^s - 2(P'_2)^s}{y^s} \frac{s - \beta_j}{s^2}$$

at $s = 0$ is equal to $-1 - \beta_j \log(y/P^{0.5})$.

Similarly, in the case $P^{0.502} < y \leq P^{0.504}/T$ we have

$$\mathcal{V}_{1j}(y) = \frac{500}{\log P} \cdot \frac{1}{2\pi i} \int_{(1)} L(1 - \beta_j + s, \chi) \left(\frac{P'_1}{y}\right)^s \frac{ds}{s^2}.$$

This yields (10.4) in a way similar to the proof of (10.3).

In the case $y \in (P^{0.5}/T, P^{0.5}] \cup (P^{0.504}/T, P^{0.504})$ and $m < T$, we have

$$\tilde{f}(\log(ym)/\log P) \ll \alpha_1,$$

so (10.5) follows by the Polya-Vinogradov inequality and partial summation; in the case $y \in (P^{0.502}/T, P^{0.502}]$, we use (10.7) and a simple estimate for the integral on the new segments to obtain (10.5). \square

Lemma 10.2. *Write*

$$\mathcal{V}_{2j}(d, r) = \sum_n \frac{\chi(n) \tilde{f}(\log(drn)/\log P) \xi_{0j}(n; d, r)}{n}.$$

If $dr \leq P^{0.5}/T$, then

$$\mathcal{V}_{2j}(d, r) = \frac{L'(1, \chi) \Pi(d, r)}{500} \beta_{j+1} \beta_{j+2} \log P + O(\mathcal{L}^{-15}); \quad (10.8)$$

if $P^{0.5} < dr \leq P^{0.502}/T$, then

$$\mathcal{V}_{2j}(d, r) = \frac{500L'(1, \chi) \Pi(d, r)}{\log P} (-1 + \mathcal{Y}_{1j}(dr)) + O(\mathcal{L}^{-15}) \quad (10.9)$$

with

$$\mathcal{Y}_{1j}(y) = (\beta_{j+1} + \beta_{j+2}) \log(y/P^{0.5}) + \frac{\beta_{j+1} \beta_{j+2}}{2} \left(\log^2 \frac{P^{0.504}}{y} - 2 \log^2 \frac{P^{0.502}}{y} \right);$$

if $P^{0.502} < dr \leq P^{0.504}/T$, then

$$\mathcal{V}_{2j}(d, r) = \frac{500L'(1, \chi) \Pi(d, r)}{\log P} (1 + \mathcal{Y}_{2j}(dr)) + O(\mathcal{L}^{-15}) \quad (10.10)$$

with

$$\mathcal{Y}_{2j}(y) = (\beta_{j+1} + \beta_{j+2}) \log(P^{0.504}/y) + \frac{\beta_{j+1} \beta_{j+2}}{2} \log^2(P^{0.504}/y);$$

if

$$dr \in (P^{0.5}/T, P^{0.5}] \cup (P^{0.502}/T, P^{0.502}] \cup (P^{0.504}/T, P^{0.504}),$$

then

$$\mathcal{V}_{2j}(d, r) \ll \mathcal{L}^{-7}. \quad (10.11)$$

Proof. First assume $dr \leq P^{0.5}/T$. In view of (8.9) and (10.6), we have

$$\begin{aligned} \mathcal{V}_{2j}(d, r) &= \frac{500}{\log P} \cdot \int_{(1)} \frac{L(1 + \beta_{j+1} + s, \chi) L(1 + \beta_{j+2} + s, \chi) \mathcal{U}_{0j}(s; d, r)}{L(1 + s, \chi)} \\ &\quad \times \frac{(P'_1)^s - 2(P'_2)^s + (P'_3)^s}{(dr)^s} \frac{ds}{s^2}. \end{aligned}$$

The contour of integration is moved in the same way as the proof of Lemma 8.1. Thus, by Lemma 5.8 and 8.2,

$$\mathcal{V}_{2j}(d, r) = \frac{500L'(1, \chi)\Pi(d, r)}{\log P} \cdot \int_{|s|=10\alpha} \frac{(\beta_{j+1} + s)(\beta_{j+2} + s)}{s} \frac{(P'_1)^s - 2(P'_2)^s + (P'_3)^s}{(dr)^s} \frac{ds}{s^2} + O(\mathcal{L}^{-15}).$$

This yields (10.8) since the function

$$\frac{(P'_1)^s - 2(P'_2)^s + (P'_3)^s}{s^2}$$

is analytic and equal to $(\log P/500)^2$ at $s = 0$.

Similarly, in the case $P^{0.5}/T < dr \leq P^{0.502}$ we have

$$\mathcal{V}_{2j}(d, r) = \frac{500L'(1, \chi)\Pi(d, r)}{\log P} \cdot \frac{1}{2\pi i} \int_{|s|=10\alpha} \frac{(\beta_{j+1} + s)(\beta_{j+2} + s)}{s} \frac{(P'_1)^s - 2(P'_2)^s}{(dr)^s} \frac{ds}{s^2} + O(\mathcal{L}^{-15});$$

in the case $P^{0.502} < dr \leq P^{0.504}/T$ we have

$$\mathcal{V}_{2j}(d, r) = \frac{500L'(1, \chi)\Pi(d, r)}{\log P} \cdot \frac{1}{2\pi i} \int_{|s|=10\alpha} \frac{(\beta_{j+1} + s)(\beta_{j+2} + s)}{s} \left(\frac{P'_1}{dr}\right)^s \frac{ds}{s^2} + O(\mathcal{L}^{-15}).$$

These yield (10.9) and (10.10). The proof of (10.11) is similar to that of (10.5) in the case $y \in (P^{0.502}/T, P^{0.502}]$. \square

By direct calculation, for $0 \leq z \leq 1$ we have

$$\beta_j \log P^{z-0.5} = \pi i j(z - 0.5) + O(\mathcal{L}^{-8}), \quad \beta_j \log P^{0.504-z} = \pi i j(0.504 - z) + O(\mathcal{L}^{-8}),$$

$$\mathcal{Y}_{1j}(P^z) = \eta_{1j}(z) + O(\mathcal{L}^{-8}), \quad \mathcal{Y}_{2j}(P^z) = \eta_{2j}(z) + O(\mathcal{L}^{-8}),$$

where

$$\eta_{11}(z) = 5\pi i(z - 0.5) - 3\pi^2((0.504 - z)^2 - 2(0.502 - z)^2),$$

$$\eta_{21}(z) = 5\pi i(0.504 - z) - 3\pi^2(0.504 - z)^2,$$

$$\eta_{12}(z) = 4\pi i(z - 0.5) - \frac{3\pi^2}{2}((0.504 - z)^2 - 2(0.502 - z)^2),$$

$$\eta_{22}(z) = 4\pi i(0.504 - z) - \frac{3\pi^2}{2}(0.504 - z)^2,$$

$$\eta_{13}(z) = 3\pi i(z - 0.5) - \pi^2((0.504 - z)^2 - 2(0.502 - z)^2),$$

$$\eta_{23}(z) = 3\pi i(0.504 - z) - \pi^2(0.504 - z)^2.$$

Remark. Similar to (10.6),

$$\tilde{f}(\log y / \log P + 0.004 - \tilde{\alpha}) = \frac{500}{\log P} \cdot \frac{1}{2\pi i} \int_{(1)} \left(\frac{Dt_0}{P^{0.004}}\right)^s \frac{(P'_1)^s - 2(P'_2)^s + (P'_3)^s}{y^s} \frac{ds}{s^2}.$$

Thus, with simple modification, Lemma 10.1 and 10.2 apply to the sums

$$\sum_m \frac{\chi(m) \tilde{f}(\log(y m) / \log P + 0.004 - \tilde{\alpha})}{m^{1-\beta_j}}$$

and

$$\sum_n \frac{\chi(n) \tilde{f}(\log(d r n) / \log P + 0.004 - \tilde{\alpha}) \xi_{0j}(n; d, r)}{n}.$$

Evaluation of $\Theta_1(\mathbf{a}_{11}, \mathbf{a}_{13})$.

We have

$$\begin{aligned} S_j(\mathbf{a}_{11}, \mathbf{a}_{13}) &= \sum_r \sum_d \frac{|\chi(d)| |\mu \chi(r)| \lambda_{0j}(dr)}{dr \varphi(r)} \left(\sum_m \frac{\chi(m) (\varkappa_1(drm) + \iota_2 \varkappa_2(drm))}{m^{1-\beta_j}} \right) \\ &\quad \times \left(\sum_n \frac{\chi(n) \tilde{f}(\log(d r n) / \log P) \xi_{0j}(n; d, r)}{n} \right). \end{aligned}$$

The right side is split into three sums according to

$$dr < P^{0.5}, \quad P^{0.5} \leq dr < P^{0.502}, \quad P^{0.502} \leq dr < P^{0.504}.$$

By Lemma 10.2 and the results in Section 8, the sum over $dr < P^{0.5}$ is equal to

$$\begin{aligned} &\frac{L'(1, \chi)^2}{500} \beta_{j+1} \beta_{j+2} \sum_{n < P^{0.5}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \left(\frac{\mathcal{F}_{j6}(P^{0.504}/n)}{0.504} + \iota_2 \frac{\mathcal{F}_{j7}(P^{0.5}/n)}{0.5} \right) + o(\alpha) \\ &= \frac{\mathbf{a} \beta_{j+1} \beta_{j+2}}{500} \int_1^{P^{0.5}} \left(\frac{\mathcal{F}_{j6}(P^{0.504}/x)}{0.504} + \iota_2 \frac{\mathcal{F}_{j7}(P^{0.5}/x)}{0.5} \right) \frac{dx}{x} + o(\alpha) \\ &= \frac{\mathbf{a} \beta_{j+1} \beta_{j+2} \log P}{500} \int_0^{0.5} \left(\frac{\mathfrak{f}_{j6}(0.004 + z)}{0.504} + \iota_2 \frac{\mathfrak{f}_{j7}(z)}{0.5} \right) dz + o(\alpha). \end{aligned}$$

Similarly, the sum over $P^{0.5} \leq dr < P^{0.502}$ is equal to

$$\begin{aligned} &\frac{500 L'(1, \chi)^2}{0.504 \log^2 P} \sum_{P^{0.5} \leq n < P^{0.502}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \mathcal{F}_{j6}(P^{0.504}/n) (-1 + \mathcal{Y}_{1j}(n)) + o(\alpha) \\ &= \frac{500 \mathbf{a}}{0.504 \log P} \int_{0.5}^{0.502} \mathfrak{f}_{j6}(0.504 - z) (-1 + \mathfrak{y}_{1j}(z)) dz + o(\alpha); \end{aligned}$$

the sum over $P^{0.502} \leq dr < P^{0.504}$ is equal to

$$\begin{aligned} &\frac{500 L'(1, \chi)^2}{0.504 \log^2 P} \sum_{P^{0.502} \leq n < P^{0.504}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \mathcal{F}_{j6}(P^{0.504}/n) (1 + \mathcal{Y}_{2j}(n)) + o(\alpha) \\ &= \frac{500 \mathbf{a}}{0.504 \log P} \int_{0.502}^{0.504} \mathfrak{f}_{j6}(0.504 - z) (1 + \mathfrak{y}_{2j}(z)) dz + o(\alpha). \end{aligned}$$

Noting that

$$\beta_{j+1}\beta_{j+2} \log P = -(11 - 6j + j^2)\pi\alpha + o(\alpha)$$

and gathering the above results together we conclude

$$\frac{1}{\alpha} S_j(\mathbf{a}_{11}, \mathbf{a}_{13}) = d_{3j}\mathbf{a} + o(1)$$

with

$$\begin{aligned} d_{3j} = & -\frac{(11 - 6j + j^2)\pi}{500} \int_0^{0.5} \left(\frac{1}{0.504} \mathfrak{f}_{j6}(0.004 + z) + \frac{\iota_2}{0.5} \mathfrak{f}_{j7}(z) \right) dz \\ & + \frac{500}{0.504\pi} \int_0^{0.002} (\mathfrak{f}_{j6}(z) - \mathfrak{f}_{j6}(0.002 + z)) dz \\ & + \frac{500}{0.504\pi} \left(\int_{0.5}^{0.502} \mathfrak{f}_{j6}(0.504 - z) \mathfrak{h}_{1j}(z) dz + \int_{0.502}^{0.504} \mathfrak{f}_{j6}(0.504 - z) \mathfrak{h}_{2j}(z) dz \right). \end{aligned}$$

Hence, by Proposition 7.1,

$$\Theta_1(\mathbf{a}_{11}, \mathbf{a}_{13}) = \left(\frac{1}{2}d_{31} + 2d_{32} + \frac{3}{2}d_{33} \right) \mathbf{a}\mathcal{P} + o(\mathcal{P}). \quad (10.12)$$

Evaluation of $\Theta_1(\mathbf{a}_{13}, \mathbf{a}_{21})$.

We have

$$\begin{aligned} S_j(\mathbf{a}_{13}, \mathbf{a}_{21}) = & \sum_r \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) \tilde{f}(\log(drm)/\log P)}{m^{1-\beta_j}} \right) \\ & \times \left(\sum_n \frac{\chi(n) (\bar{\varkappa}_1(drn) + \bar{\iota}_2 \bar{\varkappa}_2(drn)) \xi_{0j}(n; d, r)}{n} \right). \end{aligned}$$

The right side is split into three sums in the same way as in the last subsection. By Lemma 10.1 and the results in Section 8, the sum over $dr < P^{0.5}$ is $o(\alpha)$; the sum over $P^{0.5} \leq dr < P^{0.502}$ is equal to

$$\begin{aligned} & \frac{500L'(1, \chi)^2}{0.504 \log^2 P} \sum_{P^{0.5} \leq n < P^{0.502}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} (-1 - \beta_j \log(y/P^{0.5})) \mathfrak{G}_{j6}(P^{0.504}/dr) + o(\alpha) \\ & = \frac{500\mathbf{a}}{0.504 \log P} \int_{0.5}^{0.502} (-1 - \pi ij(z - 0.5)) \mathfrak{g}_{j6}(0.504 - z) dz + o(\alpha); \end{aligned}$$

the sum over $P^{0.502} < dr \leq P^{0.504}$ is equal to

$$\begin{aligned} & \frac{500L'(1, \chi)^2}{0.504 \log^2 P} \sum_{P^{0.502} \leq n < P^{0.504}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} (1 - \beta_j \log(P^{0.504}/n)) \mathfrak{G}_{j6}(P^{0.504}/n) + o(\alpha) \\ & = \frac{500\mathbf{a}}{0.504 \log P} \int_{0.502}^{0.504} (1 - \pi ij(0.504 - z)) \mathfrak{g}_{j6}(0.504 - z) dz + o(\alpha), \end{aligned}$$

Gathering these results together we conclude

$$\frac{1}{\alpha} S_j(\mathbf{a}_{13}, \mathbf{a}_{21}) = d_{4j} \mathbf{a} + o(1)$$

with

$$\begin{aligned} d_{4j} &= \frac{500}{0.504\pi} \int_0^{0.002} (\mathfrak{g}_{j6}(z) - \mathfrak{g}_{j6}(0.002 + z)) dz \\ &\quad - \frac{500ij}{0.504} \int_0^{0.002} (\mathfrak{g}_{j6}(0.002 + z)(0.002 - z) + \mathfrak{g}_{j6}(z)z) dz. \end{aligned}$$

Hence, by Proposition 7.1,

$$\Theta_1(\mathbf{a}_{13}, \mathbf{a}_{21}) = \left(\frac{1}{2} d_{41} + 2d_{42} + \frac{3}{2} d_{43} \right) \alpha \mathcal{P} + o(\mathcal{P}). \quad (10.13)$$

Evaluation of $\Theta_1(\mathbf{a}_{14}, \mathbf{a}_{22})$.

We have

$$\begin{aligned} S_j(\mathbf{a}_{14}, \mathbf{a}_{22}) &= \sum_r \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) \tilde{f}(\log(drm)/\log P + 0.004 - \tilde{\alpha})}{m^{1-\beta_j}} \right) \\ &\quad \times \left(\sum_n \frac{\chi(n) (\iota_3 \bar{\varkappa}_3(drn) + \iota_4 \bar{\varkappa}_2(drn)) \xi_{0j}(n; d, r)}{n} \right). \end{aligned}$$

The right side is split into three sums according to

$$dr < P^{0.496}, \quad P^{0.496} \leq dr < P^{0.498}, \quad P^{0.498} \leq dr < P^{0.5}.$$

By a result similar to Lemma 10.1 and the results in Section 8, the sum over $dr < P^{0.496}$ is $o(\alpha)$; the sum over $P^{0.496} \leq dr < P^{0.498}$ is equal to

$$\begin{aligned} &\frac{500L'(1, \chi)^2}{\log^2 P} \sum_{P^{0.496} \leq n < P^{0.498}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} (-1 - \beta_j \log(P^{-0.496} n)) \\ &\quad \times \left(\frac{\iota_3}{0.498} \mathcal{G}_{j6}(P^{0.498}/n) + \frac{\iota_4}{0.5} \mathcal{G}_{j7}(P^{0.5}/n) \right) + o(\alpha) \\ &= \frac{500\mathbf{a}}{\log P} \int_{0.496}^{0.498} (-1 - \pi ij(z - 0.496)) \\ &\quad \times \left(\frac{\iota_3}{0.498} \mathfrak{g}_{j6}(0.498 - z) + \frac{\iota_4}{0.5} \mathfrak{g}_{j7}(0.5 - z) \right) dz + o(\alpha); \end{aligned}$$

the sum over $P^{0.498} \leq dr < P^{0.5}$ is equal to

$$\begin{aligned} & \frac{500L'(1, \chi)^2}{\log^2 P} \frac{\iota_4}{0.5} \sum_{P^{0.498} \leq n < P^{0.5}} \frac{|\chi(n)|\lambda_{0j}(n)}{\varphi(n)} (1 - \beta_j \log(P^{0.5}/n)) \mathcal{G}_{j7}(P^{0.5}/n) + o(\alpha) \\ &= \frac{1000\iota_4 \mathbf{a}}{\log P} \int_{0.498}^{0.5} (1 - \pi i j (0.5 - z)) \mathfrak{g}_{j7}(0.5 - z) dz + o(\alpha). \end{aligned}$$

Gathering the above results together we conclude

$$\frac{1}{\alpha} S_j(\mathbf{a}_{14}, \mathbf{a}_{22}) = (d'_{5j} + d_{5j}) \mathbf{a} + o(1)$$

with

$$\begin{aligned} d'_{5j} &= -\frac{500\iota_3}{0.498\pi} \int_0^{0.002} \mathfrak{g}_{j6}(z) dz, \\ d_{5j} &= \frac{1000\iota_4}{\pi} \int_0^{0.002} (\mathfrak{g}_{j7}(z) - \mathfrak{g}_{j7}(0.002 + z)) dz \\ &\quad - \frac{500ij\iota_3}{0.498} \int_0^{0.002} (0.002 - z) \mathfrak{g}_{j6}(z) dz \\ &\quad - 1000ij\iota_4 \int_0^{0.002} ((0.002 - z) \mathfrak{g}_{j7}(0.002 + z) + z \mathfrak{g}_{j7}(z)) dz. \end{aligned}$$

Hence, by Proposition 7.1,

$$\Theta_1(\mathbf{a}_{14}, \mathbf{a}_{22}) = \left(\frac{1}{2}(d'_{51} + d_{51}) + 2(d'_{52} + d_{52}) + \frac{3}{2}(d'_{53} + d_{53}) \right) \mathbf{a} \mathcal{P} + o(\mathcal{P}). \quad (10.14)$$

Evaluation of $\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{14})$.

We have

$$\begin{aligned} S_j(\mathbf{a}_{12}, \mathbf{a}_{14}) &= \sum_r \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) (\bar{\iota}_3 \mathfrak{x}_3(drm) + \bar{\iota}_4 \mathfrak{x}_2(drm))}{m^{1-\beta_j}} \right) \\ &\quad \times \left(\sum_n \frac{\chi(n) \tilde{f}(\log(drn)/\log P + 0.004 - \tilde{\alpha}) \xi_{0j}(n; d, r)}{n} \right). \end{aligned}$$

The right side is split into three sums in the same way as in the last subsection. By a result similar to Lemma 10.2 and the results in Section 8, the sum over $dr < P^{0.496}$ is equal to

$$\begin{aligned} & \frac{L'(1, \chi)^2}{500} \beta_{j+1} \beta_{j+2} \sum_{n < P^{0.496}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \left(\frac{\bar{\iota}_3 \mathcal{F}_{j6}(P^{0.498}/n)}{0.498} + \frac{\bar{\iota}_4 \mathcal{F}_{j7}(P^{0.5}/n)}{0.5} \right) + o(\alpha) \\ &= \frac{\mathbf{a} \beta_{j+1} \beta_{j+2}}{500} \int_1^{P^{0.496}} \left(\frac{\bar{\iota}_3 \mathcal{F}_{j6}(P^{0.498}/x)}{0.498} + \frac{\bar{\iota}_4 \mathcal{F}_{j7}(P^{0.5}/x)}{0.5} \right) \frac{dx}{x} + o(\alpha) \\ &= \frac{\mathbf{a} \beta_{j+1} \beta_{j+2} \log P}{500} \int_0^{0.496} \left(\frac{\bar{\iota}_3 \mathfrak{f}_{j6}(0.002 + z)}{0.498} + \frac{\bar{\iota}_4 \mathfrak{f}_{j7}(0.004 + z)}{0.5} \right) dz + o(\alpha); \end{aligned}$$

the sum over $P^{0.496} \leq dr < P^{0.498}$ is equal to

$$\begin{aligned} & \frac{500L'(1, \chi)^2}{\log^2 P} \sum_{P^{0.496} \leq n < P^{0.498}} \frac{|\chi(n)|\lambda_{0j}(n)}{\varphi(n)} \left(\frac{\bar{l}_3 \mathcal{F}_{j6}(P^{0.498}/n)}{0.498} + \frac{\bar{l}_4 \mathcal{F}_{j7}(P^{0.5}/n)}{0.5} \right) \\ & \quad \times (-1 + \mathcal{Y}_{1j}(P^{0.004}n)) + o(\alpha) \\ & = \frac{500\mathbf{a}}{\log P} \int_0^{0.002} \left(\frac{\bar{l}_3 \mathfrak{f}_{j6}(z)}{0.498} + \frac{\bar{l}_4 \mathfrak{f}_{j7}(0.002+z)}{0.5} \right) (-1 + \mathfrak{h}_{1j}(0.502-z)) dz + o(\alpha); \end{aligned}$$

the sum over $P^{0.498} \leq dr < P^{0.5}$ is equal to

$$\begin{aligned} & \frac{1000\bar{l}_4 L'(1, \chi)^2}{\log^2 P} \sum_{P^{0.498} \leq n < P^{0.5}} \frac{|\chi(n)|\lambda_{0j}(n)}{\varphi(n)} \mathcal{F}_{j7}(P^{0.5}/n) (1 + \mathcal{Y}_{2j}(P^{0.004}n)) + o(\alpha) \\ & = \frac{1000\bar{l}_4 \mathbf{a}}{\log P} \int_0^{0.002} \mathfrak{f}_{j7}(z) (1 + \mathfrak{h}_{2j}(0.504-z)) dz + o(\alpha). \end{aligned}$$

Gathering the above results together we conclude

$$\frac{1}{\alpha} S_j(\mathbf{a}_{12}, \mathbf{a}_{14}) = (d'_{6j} + d_{6j})\mathbf{a} + o(1)$$

with

$$\begin{aligned} d'_{6j} &= -\frac{500\bar{l}_3}{0.498\pi} \int_0^{0.002} \mathfrak{f}_{j6}(z) dz, \tag{10.15} \\ d_{6j} &= -\frac{(11-6j+j^2)\pi}{500} \int_0^{0.496} \left(\frac{\bar{l}_3 \mathfrak{f}_{j6}(0.002+z)}{0.498} + \frac{\bar{l}_4 \mathfrak{f}_{j7}(0.004+z)}{0.5} \right) dz \\ & \quad + \frac{1000\bar{l}_4}{\pi} \int_0^{0.002} (\mathfrak{f}_{j7}(z) - \mathfrak{f}_{j7}(0.002+z)) dz \\ & \quad + \frac{500}{\pi} \int_0^{0.002} \left(\frac{\bar{l}_3 \mathfrak{f}_{j6}(z)}{0.498} + \frac{\bar{l}_4 \mathfrak{f}_{j7}(0.002+z)}{0.5} \right) \mathfrak{h}_{1j}(0.502-z) dz \\ & \quad + \frac{1000\bar{l}_4}{\pi} \int_0^{0.002} \mathfrak{f}_{j7}(z) \mathfrak{h}_{2j}(0.504-z) dz. \end{aligned}$$

Hence, by Proposition 7.1,

$$\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{14}) = \left(\frac{1}{2}(d'_{61} + d_{61}) + 2(d'_{62} + d_{62}) + \frac{3}{2}d'_{63} + d_{63} \right) \mathbf{a}\mathcal{P} + o(\mathcal{P}). \tag{10.16}$$

It follows from (10.1) and (10.12)-(10.16) that

$$\Xi_1^* = (\mathfrak{d}' + \mathfrak{d})\mathbf{a}\mathcal{P} + o(\mathcal{P}) \tag{10.17}$$

where

$$\mathfrak{d}' = \frac{1}{2}(d'_{51} + \overline{d'_{61}}) + 2(d'_{52} + \overline{d'_{62}}) + \frac{3}{2}(d'_{53} + \overline{d'_{63}}),$$

$$\mathfrak{d} = \frac{1}{2}(d_{31} + d_{51} + \overline{d_{41} + d_{61}}) + 2(d_{32} + d_{52} + \overline{d_{42} + d_{62}}) + \frac{3}{2}(d_{33} + d_{53} + \overline{d_{43} + d_{63}})$$

For $\mu = 6, 7$,

$$\mathfrak{f}_{j\mu}(0) = \mathfrak{g}_{j\mu}(0) = 1.$$

Hence

$$\mathfrak{d}' \simeq -\frac{8\iota_3}{0.498\pi}.$$

We only need a crude lower bound for the real part of \mathfrak{d}' . By direct calculation we have

$$\Re\{\mathfrak{d}'\} > -\frac{8}{0.498\pi}\Re\{\iota_3\} - 0.04 > 5.1.$$

The contribution from \mathfrak{d} is minor since if z is close to $1/2$, then for $\mu = 1, 2$,

$$\mathfrak{h}_{\mu j}(z) \simeq 0.$$

Thus, by direct calculation we have the crude bound

$$|\Re\{\mathfrak{d}\}| < 0.1.$$

Combining these bounds with (10.16) we complete the proof of Proposition 2.4. \square

Remark. Numerical calculation actually shows that

$$\Re\{\mathfrak{d}\} > 0.$$

11. Proof of Proposition 2.6

By the result of Section 9,

$$\Xi_{12} \ll \mathfrak{a}\mathcal{P}.$$

Hence, by Cauchy's inequality, the proof of Proposition 2.6 is reduced to showing that

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |J_1(\rho, \psi) - Z(\rho, \chi\psi) J_2(1 - \rho, \bar{\psi})|^2 \omega(\rho) = o(\mathfrak{a}\mathcal{P}). \quad (11.1)$$

Let $\tilde{g}_1(y)$ and $\tilde{g}_2(y)$ be given by

$$\begin{aligned} \tilde{g}_1(y) &= -500 \int_{0.5}^{0.502} g\left(\frac{Pz}{y}\right) dz + 500 \int_{0.502}^{0.504} g\left(\frac{Pz}{y}\right) dz, \\ \tilde{g}_2(y) &= -500 \int_{0.496}^{0.498} g\left(\frac{PzDt_0}{y}\right) dz + 500 \int_{0.498}^{0.5} g\left(\frac{PzDt_0}{y}\right) dz. \end{aligned}$$

Write

$$\tilde{J}_\mu(s, \psi) = \sum_n \frac{\chi\psi(n)\tilde{g}_\mu(n)}{n^s}, \quad \mu = 1, 2,$$

and

$$\eta_\pm = \exp\{\pm\mathcal{L}^{-10}\}.$$

By (5) and (5), for $\sigma = 1/2$, the terms with $n \leq P^{0.5}\eta_-$ or $n \geq P^{0.504}\eta_+$ in $\tilde{J}_1(s, \psi)$ contribute $\ll \varepsilon$.

Lemma 11.1. *If*

$$y \in [P^{0.5}\eta_+, P^{0.502}\eta_-] \cup [P^{0.502}\eta_+, P^{0.504}\eta_-],$$

then

$$\tilde{f}\left(\frac{\log y}{\log P}\right) - \tilde{g}_1(y) \ll \varepsilon; \quad (11.2)$$

if

$$y \in (P^{0.5}\eta_-, P^{0.5}\eta_+) \cup (P^{0.502}\eta_-, P^{0.502}\eta_+) \cup (P^{0.504}\eta_-, P^{0.504}\eta_+),$$

then

$$\tilde{f}\left(\frac{\log y}{\log P}\right) - \tilde{g}_1(y) \ll \mathcal{L}^{-10}. \quad (11.3)$$

Proof. Since

$$g(1/x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\mathcal{L}\log x} \exp(-t^2) dt = \frac{1}{\sqrt{\pi}} \int_{\mathcal{L}\log x}^{\infty} \exp(-t^2) dt,$$

it follows that

$$g(x) + g(1/x) = 1. \quad (11.4)$$

Write

$$u = \frac{\log y}{\log P}.$$

First assume $T^{0.502}\eta_+ \leq y \leq P^{0.503}$. By (11.4),

$$\int_{0.502}^{2u-0.502} g\left(\frac{Pz}{y}\right) dz = u - 0.502;$$

by (6.2),

$$\int_{2u-0.502}^{0.504} g\left(\frac{Pz}{y}\right) dz = 1.006 - 2u + O(\varepsilon);$$

by (6.3),

$$\int_{0.5}^{0.502} g\left(\frac{Pz}{y}\right) dz \ll \varepsilon.$$

These together imply (11.2).

Now assume $P^{0.503} \leq y \leq P^{0.504}\eta_-$. By (11.4),

$$\int_{2u-0.504}^{0.504} g\left(\frac{P^z}{y}\right) dz = 0.504 - u;$$

by (6.2),

$$\int_{0.502}^{2u-0.504} g\left(\frac{P^z}{y}\right) dz = 2u - 1.006 + O(\varepsilon);$$

by (6.3),

$$\int_{0.5}^{0.502} g\left(\frac{P^z}{y}\right) dz \ll \varepsilon.$$

These together imply (11.2). The proof of (11.2) with $y \in [P^{0.5}\eta_+, P^{0.502}\eta_-]$ is similar.

We give the proof of (11.3) with $P^{0.502}\eta_- < y < P^{0.502}\eta_+$ only; the other cases are similar. We have

$$\int_{0.502}^{0.504} g\left(\frac{P^z}{y}\right) dz = \frac{1}{500} + O(\mathcal{L}^{-10}), \quad \int_{0.5}^{0.502} g\left(\frac{P^z}{y}\right) dz = O(\mathcal{L}^{-10}),$$

by (5.2) and (5.3), and

$$\tilde{f}\left(\frac{\log y}{\log P}\right) = 1 + O(\mathcal{L}^{-10}). \quad \square$$

It follows from Lemma 11.1 that

$$J_1(s, \psi) - \tilde{J}_1(s, \psi) = \sum_{n \in \mathfrak{I}_1} \frac{\chi\psi(n)}{n^s} (\tilde{f}(\log n / \log P) - \tilde{g}_1(n)) + O(\varepsilon) \quad (11.5)$$

for $\sigma = 1/2$, where

$$\mathfrak{I}_1 = (P^{0.5}\eta_-, P^{0.5}\eta_+) \cup (P^{0.502}\eta_-, P^{0.502}\eta_+) \cup (P^{0.504}\eta_-, P^{0.504}\eta_+).$$

By (11.3), (8.25) and (8.26),

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) \left| \sum_{n \in \mathfrak{I}_1} \frac{\chi\psi(n)}{n^\rho} (\tilde{f}(\log n / \log P) - \tilde{g}_1(n)) \right|^2 \omega(\rho) = o(\mathfrak{aP}).$$

Hence

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |J_1(\rho, \psi) - \tilde{J}_1(\rho, \psi)|^2 \omega(\rho) = o(\mathfrak{aP}).$$

Similarly,

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |J_2(\rho, \psi) - \tilde{J}_2(\rho, \psi)|^2 \omega(\rho) = o(\mathfrak{aP}).$$

The proof of (11.1) is therefore reduced to showing that

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |\tilde{J}_1(\rho, \psi) - Z(\rho, \chi\psi) \tilde{J}_2(1 - \rho, \bar{\psi})|^2 \omega(\rho) = o(\mathfrak{aP}). \quad (11.6)$$

Lemma 11.2. *Suppose $\sigma = 1/2$ and $|t - 2\pi t_0| < \mathcal{L}_1$. Then*

$$\tilde{J}_1(s, \psi) = Z(s, \chi\psi) \tilde{J}_2(1 - s, \bar{\psi}) + O(E_2(s, \psi))$$

where

$$E(s, \psi) = \mathcal{L}^{-68} \int_{-\mathcal{L}^{20}}^{\mathcal{L}^{20}} \left| \sum_{n < P_1} \frac{\chi\psi(n)}{n^{s+iv}} \right| \omega_1(iv) dv.$$

Proof. Suppose $0.5 \leq z \leq 0.504$. In a way similar to the proof of Lemma 6.1, it can be deduced that

$$\sum_n \frac{\chi\psi(n)}{n^s} g\left(\frac{P^z}{n}\right) = L(s, \chi\psi) - Z(s, \chi\psi) \sum_n \frac{\chi\bar{\psi}(n)}{n^{1-s}} g\left(\frac{P^{1-z}Dt_0}{n}\right) + O(E_2(s, \psi)).$$

Hence

$$\begin{aligned} & \int_{0.5}^{0.502} \left\{ \sum_n \frac{\chi\psi(n)}{n^s} g\left(\frac{P^z}{n}\right) \right\} dz \\ &= \frac{1}{500} L(s, \chi\psi) - Z(s, \chi\psi) \int_{0.498}^{0.5} \left\{ \sum_n \frac{\chi\bar{\psi}(n)}{n^{1-s}} g\left(\frac{P^zDt_0}{n}\right) \right\} dz + O(E_2(s, \psi)), \end{aligned}$$

and

$$\begin{aligned} & \int_{0.502}^{0.504} \left\{ \sum_n \frac{\chi\psi(n)}{n^s} g\left(\frac{P^z}{n}\right) \right\} dz \\ &= \frac{1}{500} L(s, \chi\psi) - Z(s, \chi\psi) \int_{0.496}^{0.498} \left\{ \sum_n \frac{\chi\bar{\psi}(n)}{n^{1-s}} g\left(\frac{P^zDt_0}{n}\right) \right\} dz + O(E_2(s, \psi)). \end{aligned}$$

This completes the proof. \square

By Lemma 11.2, the proof of (11.6) is reduced to showing that

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) E_2(\rho, \psi)^2 \omega(\rho) = o(\mathfrak{aP}).$$

This follows by (8.25), (8.26) and simple estimates.

12. Evaluation of Ξ_{15}

The sum $H_{11}(s, \psi)$ is split into

$$H_{11}(s, \psi) = \sum_{n < P^{1/2}} + \sum_{P^{1/2} \leq n < P_1} = H_{14}(s, \psi) + H_{15}(s, \psi), \quad \text{say.} \quad (12.1)$$

We can write

$$H_1(s, \psi)H_2(s, \psi) = B(s, \psi) + H_{15}(s, \psi)H_2(s, \psi)$$

with

$$B(s, \psi) = (H_{14}(s, \psi) + \iota_2 H_{12}(s, \psi))H_2(s, \psi). \quad (12.2)$$

Accordingly we have

$$\Xi_{13} = \Xi_{14} + \Xi_{15} \quad (12.3)$$

with

$$\Xi_{14} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) Z(\rho, \chi\psi)^{-1} B(\rho, \psi) \omega(\rho), \quad (12.4)$$

$$\Xi_{15} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) Z(\rho, \chi\psi)^{-1} H_{15}(\rho, \psi) H_2(\rho, \psi) \omega(\rho). \quad (12.5)$$

The goal of this section is to evaluate Ξ_{15} .

Write

$$\varkappa_{12}(y) = \frac{1}{0.504} \left(\frac{P_1}{y} \right)^{\beta_6} \int_{0.5}^{0.504} \left\{ g\left(\frac{Pz}{y} \right) - g\left(\frac{P^{0.5}}{y} \right) \right\} dz$$

and

$$\tilde{H}_{15}(s, \psi) = \sum_{P^{0.5}\eta_- < n < P_1\eta_+} \frac{\varkappa_{12}(n) \chi\psi(n)}{n^s}.$$

We first claim that

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |H_{15}(\rho, \psi) - \tilde{H}_{15}(\rho, \psi)|^2 \omega(\rho) = o(\mathfrak{aP}). \quad (12.6)$$

By the argument in the last section, for $P^{0.5}\eta_+ < n < P_1\eta_-$,

$$\int_{0.5}^{0.504} g\left(\frac{Pz}{y} \right) dz = \frac{\log P_1 - \log n}{\log P} + O(\varepsilon),$$

so that

$$\varkappa_{12}(n) - \varkappa_1(n) \ll \varepsilon.$$

Also we have

$$\varkappa_{12}(n) - \varkappa_1(n) \ll \begin{cases} \mathcal{L}^{-10} & \text{if } P_1\eta_- \leq n < P_1\eta_+, \\ 1 & \text{if } P^{0.5}\eta_- < n \leq P^{0.5}\eta_+. \end{cases}$$

These bounds together with (8.25) and (8.26) imply (12.6). We briefly describe the argument as follows. Let $\mathbf{t}(y)$ denote the characteristic function of the interval $(P^{0.5}\eta_-, P^{0.5}\eta_+]$. If $d \leq P^{0.5}(\eta_+ - \eta_-)$, then

$$\sum_l \frac{\mathbf{t}(dl)}{l} \ll \mathcal{L}^{-10}.$$

If $d > P^{0.5}(\eta_+ - \eta_-)$, then there exists at most one l such that $\mathbf{t}(dl) = 1$. Hence

$$\sum_{d > P^{0.5}(\eta_+ - \eta_-)} \frac{1}{d} \left(\sum_l \frac{\mathbf{t}(dl)}{l} \right) \leq \sum_l \frac{1}{l^2} \sum_d \frac{\mathbf{t}(dl)}{d} \ll \mathcal{L}^{-10}.$$

Assume $\sigma = 1/2$, $|t - 2\pi t_0| < \mathcal{L}_1$ and $0.5 \leq z \leq 0.504$. Using the proof of Lemma 11.2 with $s + \beta_6$ in place of s we deduce that

$$\begin{aligned} \tilde{H}_{15}(s, \psi) &= \frac{Z(s + \beta_6, \chi\psi)P_1^{\beta_6}}{0.504} \sum_n \frac{\chi\bar{\psi}(n)}{n^{1-s-\beta_6}} \int_{0.496}^{0.5} \left\{ g\left(\frac{P^{0.5}Dt_0}{n}\right) - g\left(\frac{P^zDt_0}{n}\right) \right\} dz \\ &\quad + O(E(s + \beta_6, \psi)). \end{aligned}$$

Write

$$P_1'' = P^{0.496}Dt_0, \quad P_2'' = P^{0.5}Dt_0.$$

By (4) and (4), the terms with $n \leq P_1''\eta_-$ or $n \geq P_2''\eta_+$ above contribute $\ll \varepsilon$. If

$$P_1''\eta_+ < n < P_2''\eta_-,$$

then, in a way similar to the proof of (10.),

$$\int_{0.496}^{0.5} \left\{ g\left(\frac{P^{0.5}Dt_0}{n}\right) - g\left(\frac{P^zDt_0}{n}\right) \right\} dz = \frac{\log n}{\log P} - \tilde{\alpha} - 0.496 + O(\varepsilon).$$

Thus, in a way similar to the proof of Proposition 2.6, by the above discussion we deduce that

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) |Z(\rho, \chi\psi)^{-1} \tilde{H}_{15}(\rho, \psi) - \bar{H}_{16}(1 - s, \bar{\psi})|^2 \omega(\rho) = o(\mathfrak{aP}). \quad (12.8)$$

where

$$\bar{H}_{16}(1 - s, \bar{\psi}) = \frac{1}{\log P_1} \sum_{P_1'' < n < P_2''} \frac{\chi\bar{\psi}(n)}{n^{1-s}} (n/P_1'')^{\beta_6} \log(n/P_1'').$$

By (12.7) and (12.8) we obtain

$$\Xi_{15} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \tilde{Z}(\psi)} \mathcal{C}^*(\rho, \psi) \bar{H}_{16}(1 - \rho, \bar{\psi}) H_2(\rho, \psi) \omega(\rho) + o(\mathcal{P}).$$

This implies, by Lemma 8.1,

$$\Xi_{15} = \Theta_1(\mathbf{a}_{12}, \mathbf{a}_{25}) + \overline{\Theta_1(\mathbf{a}_{15}, \mathbf{a}_{22})} + o(\mathcal{P}) \quad (12.9)$$

where

$$a_{15}(n) = \chi(n) \varkappa_{13}(n)$$

with

$$\varkappa_{13}(n) = \begin{cases} (\log P_1)^{-1} (n/P_1'')^{-\beta_6} \log(n/P_1'') & \text{if } P_1'' < n < P_2'', \\ 0 & \text{otherwise,} \end{cases}$$

$$a_{25}(n) = \overline{a_{15}(n)}.$$

Assume $1 \leq j \leq 3$ in what follows.

Lemma 12.1. *We have*

$$\sum_l \frac{\chi(l) \varkappa_{13}(dl)}{l^{1-\beta_j}} \ll \begin{cases} T^{-c} & \text{if } d \leq P_1''/T, \\ \alpha_1 & \text{if } P_1''/T < d \leq P_1'', \end{cases}$$

and

$$\sum_l \frac{\chi(l) \varkappa_{13}(dl)}{l^{1-\beta_j}} = \frac{L'(1, \chi)}{\log P_1} (-1 + (2\beta_6 - \beta_j) \log(d/P_1'') + \epsilon_{1j}(d)), \quad |\epsilon_{1j}(d)| < 10^{-5},$$

if $P_1'' < d < P_2''$.

Proof. Note that $\varkappa_{13}(n) \ll \alpha_1$ if $P_1''/T \leq n \leq P_1''T$. In the case $d \leq P_1''$ the results follow by the Polya-Vinogradov inequality and partial summation. On the other hand, in the case $P_1'' < d < P_2''$ the sum becomes

$$\begin{aligned} & \frac{(d/P_1'')^{-\beta_6} \log(d/P_1'')}{\log P_1} \sum_{l < P_2''/d} \frac{\chi(l)}{l^{1+\beta_6-\beta_j}} + \frac{(d/P_1'')^{-\beta_6}}{\log P_1} \sum_{l < P_2''/d} \frac{\chi(l) \log l}{l^{1+\beta_6-\beta_j}} + O(T^{-c}) \\ & = \frac{L'(1, \chi)(d/P_1'')^{-\beta_6}}{\log P_1} (-1 + (\beta_6 - \beta_j) \log(d/P_1'')) + O(\mathcal{L}^{-15}). \end{aligned}$$

Since $P_2''/d > T^{10}$ and $(d/P_1'')^{-\beta_6} = 1 - \beta_6 \log(d/P_1'') + \dots$, the result follows by Lemma 5.8 and a simple estimates. \square

Lemma 12.2. *If $dr < P_1''/T$, then*

$$\sum_l \frac{\chi(l) \bar{\alpha}_{13}(drl) \xi_j(l; d, r)}{l} = b^* L'(1, \chi) \Pi(d, r) (\log P) \beta_{j+1} \beta_{j+2} + O \quad (12.10)$$

with

$$b^* = \frac{1}{0.504} \int_0^{0.004} z e^{3\pi i z/2} dz.$$

If $P_1''/T \leq n \leq P_1''$, then

$$\sum_l \frac{\chi(l) \bar{\alpha}_{13}(drl) \xi_j(l; d, r)}{l} \ll \alpha_1; \quad (12.11)$$

Proof. If $P_1'' < n < P_2''$, then

$$\bar{\alpha}_{13}(n) = -\frac{1}{\log P_1} \frac{\partial}{\partial w} (n/P_1'')^{\beta_6-w} \Big|_{w=0}.$$

Hence

$$\sum_l \frac{\chi(l) \bar{\alpha}_{13}(drl) \xi_j(l; d, r)}{l} = -\frac{1}{\log P_1} \frac{\partial}{\partial w} \left((dr/P_1'')^{\beta_6-w} \sum_{P_1''/dr < l < P_2''/dr} \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w}} \right) \Big|_{w=0}.$$

Suppose $dr \leq P_1''/T$. By (4) and (4), for $|w| = \alpha$ we have

$$\begin{aligned} \sum_{P_1''/dr < l < P_2''/dr} \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w}} &= \sum_l \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w}} \left\{ g\left(\frac{P_2''}{drl}\right) - g\left(\frac{P_1''}{drl}\right) \right\} + O(\mathcal{L}^{-15}) \\ &= \frac{1}{2\pi i} \int_{(1)} \left(\sum_l \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w+s}} \right) \left((P_2''/dr)^s - (P_1''/dr)^s \right) \frac{\omega_1(s) ds}{s} + O(\mathcal{L}^{-15}). \end{aligned}$$

In a way similar to the proof of Lemma 8.4, by lemma 8.2 and 5.8, we find that the right side above is equal to

$$\begin{aligned} L'(1, \chi) \Pi(d, r) \cdot \frac{1}{2\pi i} \int_{|s|=5\alpha} \frac{(s+w-\beta_6+\beta_{j+1})(s+w-\beta_6+\beta_{j+2})}{s+w-\beta_6} \frac{(P_2''/dr)^s - (P_1''/dr)^s}{s} ds \\ + O(\mathcal{L}^{-15}) \\ = L'(1, \chi) \Pi(d, r) \beta_{j+1} \beta_{j+2} \frac{(P_2''/dr)^{\beta_6-w} - (P_1''/dr)^{\beta_6-w}}{\beta_6-w} + O(\mathcal{L}^{-15}). \end{aligned}$$

It follows by Cauchy' integral formula that

$$\frac{\partial}{\partial w} \left((dr/P_1'')^{\beta_6-w} \sum_{P_1''/dr < l < P_2''/dr} \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w}} \right) \Big|_{w=0} = \frac{\partial}{\partial w} \left(\int_1^{P_2''/dr} y^{\beta_6-w-1} dy \right) \Big|_{w=0} + O(\mathcal{L}^{-6})$$

The main term on the right side is, by substituting $y = P^z$, equal to

$$-(\log P)^2 \int_0^{0.004} z e^{3\pi iz/2} dz.$$

Gathering these results together we obtain (12.10). The proof of (12.11) is similar to that of .

Lemma 12.3. *If $P_1'' < dr < P_2$, then*

$$\sum_l \frac{\chi(l) \bar{\zeta}_{13}(drl) \xi_j(l; d, r)}{l} = \frac{L'(1, \chi) \Pi(d, r)}{\log P_1} \left(-1 + (-2\beta_6 + \beta_{j+1} + \beta_{j+2}) \log(dr/P_1'') + \epsilon_{2j}(dr) \right),$$

$$|\epsilon_{2j}(dr)| < 10^{-5}.$$

Proof. The left side is equal to

$$-\frac{1}{\log P_1} \frac{\partial}{\partial w} \left((dr/P_1'')^{\beta_6-w} \sum_{l < P_2''/dr} \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w}} \right) \Big|_{w=0}.$$

Assume $|w| = \alpha$. In a way similar to the proof of Lemma 12.1, we deduce that

$$\sum_{l < P_2''/dr} \frac{\chi(l) \xi_j(l; d, r)}{l^{1-\beta_6+w}} = L'(1, \chi) \Pi(d, r)$$

$$\times \frac{1}{2\pi i} \int_{|s|=5\alpha} \frac{(s+w-\beta_6+\beta_{j+1})(s+w-\beta_6+\beta_{j+2})}{s+w-\beta_6} \frac{(P_2''/dr)^s}{s} ds + O(\mathcal{L}^{-15}).$$

By direct calculation,

$$\frac{1}{2\pi i} \int_{|s|=5\alpha} \frac{(s+w-\beta_6+\beta_{j+1})(s+w-\beta_6+\beta_{j+2})}{s+w-\beta_6} \frac{(P_2''/dr)^s}{s} ds$$

$$= w - \beta_6 + \beta_{j+1} + \beta_{j+2} + \beta_{j+1}\beta_{j+2} \int_1^{P_2''/dr} y^{\beta_6-w-1} dy,$$

and the derivative of

$$(dr/P_1'')^{\beta_6-w} \left(w - \beta_6 + \beta_{j+1} + \beta_{j+2} + \beta_{j+1}\beta_{j+2} \int_1^{P_2''/dr} y^{\beta_6-w-1} dy \right)$$

at $w = 0$ is equal to

$$(dr/P_1'')^{\beta_6} \left(1 - \beta_{j+1}\beta_{j+2} \int_1^{P_2''/dr} y^{\beta_6-1} \log y dy \right)$$

$$- (dr/P_1'')^{\beta_6} \log(dr/P_1'') \left(-\beta_6 + \beta_{j+1} + \beta_{j+2} + \beta_{j+1}\beta_{j+2} \int_1^{P_2''/dr} y^{\beta_6-1} dy \right).$$

This can be written as the form

$$1 - (-2\beta_6 + \beta_{j+1} + \beta_{j+2}) \log(dr/P_1'') - \epsilon_{2j}(dr),$$

since $(dr/P_1'')^{\beta_6} = 1 + \beta_6 \log(dr/P_1'') + \dots$ \square

Evaluation of $\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{25})$.

We have

$$S_j(\mathbf{a}_{12}, \mathbf{a}_{25}) = \sum_r \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) (\bar{t}_3 \varkappa_3(drm) + \bar{t}_4 \varkappa_2(drm))}{m^{1-\beta_j}} \right) \\ \times \left(\sum_n \frac{\chi(n) \bar{\varkappa}_{13}(n) \xi_{0j}(n; d, r)}{n} \right).$$

The sum is split into three sums according to $dr \leq P_1''/T$, $P_1''/T < dr \leq P_1''$ and $P_1'' < dr < P_2$. By lemma 8.2, 8.3 and 12.1, the sum over $P_1''/T < dr \leq P_1''$ contributes $o(\alpha)$; the sum over $dr \leq P_1''/T$ is equal to

$$L'(1, \chi)^2 b^* \beta_{j+1} \beta_{j+2} \sum_{n < P^{0.496}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \left(\frac{\bar{t}_3 \mathcal{F}_{j6}(P^{0.498}/n)}{0.498} + \frac{\bar{t}_4 \mathcal{F}_{j7}(P^{0.5}/n)}{0.5} \right) + o(\alpha) \\ = \mathbf{a} b^* (\log P) \beta_{j+1} \beta_{j+2} \int_0^{0.496} \left(\frac{\bar{t}_3 \mathfrak{f}_{j6}(0.498 - z)}{0.498} + \frac{\bar{t}_4 \mathfrak{f}_{j7}(0.5 - z)}{0.5} \right) dz + o(\alpha). \quad (12.12)$$

By lemma 8.2, 8.3 and 12.3, the sum over $P_1'' < dr < P_2$ is equal to

$$\frac{L'(1, \chi)^2 \bar{t}_3}{(0.504)(0.498) \log^2 P} \sum_{P^{0.496} < n < P^{0.498}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \mathcal{F}_{j6}(P^{0.498}/n) \mathcal{W}_j(n) \\ + \frac{L'(1, \chi)^2 \bar{t}_4}{(0.504)(0.5) \log^2 P} \sum_{P^{0.496} < n < P^{0.5}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \mathcal{F}_{j7}(P^{0.5}/n) \mathcal{W}_j(n) + o(\alpha) \quad (12.13) \\ = \frac{\mathbf{a} \bar{t}_3}{(0.504)(0.498) \log P} \int_{0.496}^{0.498} \mathfrak{f}_{j6}(0.498 - z) \mathcal{W}_j(P^z) dz \\ + \frac{\mathbf{a} \bar{t}_4}{(0.504)(0.5) \log P} \int_{0.496}^{0.5} \mathfrak{f}_{j7}(0.5 - z) \mathcal{W}_j(P^z) dz + o(\alpha).$$

where

$$\mathcal{W}_j(n) = -1 + (-2\beta_6 + \beta_{j+1} + \beta_{j+2}) \log(n/P_1'') + \epsilon_{2j}(n).$$

For $0 \leq z \leq 0.002$, a good approximation to $\mathfrak{f}_{j6}(z)$ is

$$1 + \pi i(3 - j)z \simeq 1 + (2\beta_6 - \beta_j)(\log P)z.$$

Since

$$4\beta_6 - \beta_1 - \beta_2 - \beta_3 \simeq 0,$$

it follows by simple calculation that

$$\int_{0.496}^{0.498} \mathfrak{f}_{j6}(0.498 - z)\mathcal{W}_j(P^z) dz = -0.002 + \epsilon/10.$$

For $0 \leq z \leq 0.004$, a good approximation to $\mathfrak{f}_{j7}(z)$ is

$$1 + \pi i(5 - j)z \simeq 1 + (2\beta_7 - \beta_j)(\log P)z.$$

Thus, for $0.496 \leq z \leq 0.5$, the function $\mathfrak{f}_{j7}(0.5 - z)\mathcal{W}_j(P^z)$ can be well approximated by

$$-1 - (2\beta_7 - \beta_j)(\log P)(0.5 - z) - (2\beta_6 - \beta_{j+1} - \beta_{j+2})(\log P)(z - 0.496).$$

Since

$$(2\beta_6 + 2\beta_7 - \beta_1 - \beta_2 - \beta_3) \log P \simeq 2\pi i,$$

it follows that

$$\int_{0.496}^{0.5} \mathfrak{f}_{j6}(0.5 - z)\mathcal{W}_j(P^z) dz = -0.004 - \frac{\pi i}{250^2} + \epsilon/10.$$

inserting these results into (12.13), we find that the sum over $P_1'' < dr < P_2$ is equal to

$$\frac{\mathbf{a}}{0.504 \log P} \left(-\frac{0.002\bar{\iota}_3}{0.498} - 0.008\bar{\iota}_4 - \frac{2\pi i\bar{\iota}_4}{250^2} + \epsilon/4 \right). \quad (12.14)$$

It follows from (12.12) and (12.14) that

$$\frac{1}{2\alpha} S_1(\mathbf{a}_{12}, \mathbf{a}_{25}) + \frac{2}{\alpha} S_2(\mathbf{a}_{12}, \mathbf{a}_{25}) + \frac{3}{2\alpha} S_3(\mathbf{a}_{12}, \mathbf{a}_{25}) = \mathbf{a}(e_1^* + e_2^* + \epsilon/2) \quad (12.15)$$

where

$$e_1^* = -\pi b^*(3e_{11}^* + 6e_{12}^* + 3e_{13}^*)$$

with

$$e_{1j}^* = \int_0^{0.496} \left(\frac{\bar{\iota}_3 \mathfrak{f}_{j6}(0.498 - z)}{0.498} + \frac{\bar{\iota}_4 \mathfrak{f}_{j7}(0.5 - z)}{0.5} \right) dz,$$

and

$$e_2^* = \frac{4}{0.504\pi} \left(-\frac{0.002\bar{\iota}_3}{0.498} - 0.008\bar{\iota}_4 - \frac{2\pi i\bar{\iota}_4}{250^2} \right).$$

Evaluation of $\Theta_1(\mathbf{a}_{15}, \mathbf{a}_{22})$.

We have

$$S_j(\mathbf{a}_{15}, \mathbf{a}_{22}) = \sum_r \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) \varkappa_{13}(m)}{m^{1-\beta_j}} \right) \\ \times \left(\sum_n \frac{\chi(n) (\iota_3 \bar{\varkappa}_3(drn) + \iota_4 \bar{\varkappa}_2(drn)) \xi_{0j}(n; d, r)}{n} \right).$$

The sum is split into two sums according to $dr \leq P_1''$ and $P_1'' < dr < P_2$. By lemma 8.2, 8.3 and 12.1, the first sum contributes $o(\alpha)$; the second sum is equal to

$$\mathbf{a}b^* \int_0^{0.496} (\bar{\iota}_3 \mathbf{f}_j 6(0.498 - z) + \bar{\iota}_4 \mathbf{f}_j 6(0.5 - z)) dz + o(\alpha).$$

By lemma 8.2, 8.3 and 12.3, the sum over $P_1'' < dr < P_2$ is equal to

$$\frac{L'(1, \chi)^2 \iota_3}{(0.504)(0.498) \log^2 P} \sum_{P^{0.496} < n < P^{0.498}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \mathcal{G}_{j6}(P^{0.498}/n) \mathcal{W}_j^*(n) \\ + \frac{L'(1, \chi)^2 \iota_4}{(0.504)(0.5) \log^2 P} \sum_{P^{0.496} < n < P^{0.5}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} \mathcal{G}_{j7}(P^{0.5}/n) \mathcal{W}_j^*(n) + o(\alpha) \\ = \frac{\mathbf{a}\iota_3}{(0.504)(0.498) \log P} \int_{0.496}^{0.498} \mathbf{g}_{j6}(0.498 - z) \mathcal{W}_j^*(P^z) dz \\ + \frac{\mathbf{a}\iota_4}{(0.504)(0.5) \log P} \int_{0.496}^{0.5} \mathbf{g}_{j7}(0.5 - z) \mathcal{W}_j^*(P^z) dz + o(\alpha).$$

where

$$\mathcal{W}^*(n) = -1 + (2\beta_6 - \beta_j) \log(n/P_1'') + \epsilon_{1j}(n).$$

For $0 \leq z \leq 0.004$ and $\mu = 6, 7$, the function $\mathbf{g}_{j\mu}(z)$ is well-approximated by

$$1 - (2\beta_\mu - \beta_{j+1} - \beta_{j+2})(\log P)z.$$

Thus, in a way similar to the evaluation of $\Theta_1(\mathbf{a}_{12}, \mathbf{a}_{25})$, we deduce that

$$\frac{1}{2\alpha} S_1(\mathbf{a}_{15}, \mathbf{a}_{23}) + \frac{2}{\alpha} S_2(\mathbf{a}_{15}, \mathbf{a}_{22}) + \frac{3}{2\alpha} S_3(\mathbf{a}_{15}, \mathbf{a}_{22}) = \mathbf{a}(\bar{e}_2^* + \epsilon/2) \quad (12.16).$$

Finally, by (12.9), (12.15) and (12.16) we conclude

$$\Xi_{15} = (e_1^* + 2e_2^* + \epsilon)\mathbf{a}\mathcal{P}. \quad (12.17)$$

13. Approximation to Ξ_{14}

In this section we establish an approximation to Ξ_{14} .

Assume that $\psi \in \Psi_1$ and $\rho \in \mathcal{Z}(\psi)$. By Lemma 5.2 and (2.2),

$$\begin{aligned} \mathcal{C}^*(\rho, \psi) &= -iZ(\rho + \beta_3, \psi)^{-1} \frac{L(\rho + \beta_1, \psi)L(\rho + \beta_2, \psi)L(\rho + \beta_3, \psi)}{L'(\rho, \psi)} (1 + O(1/t_0)) \\ &= -i \frac{L(\rho + \beta_1, \psi)L(\rho + \beta_2, \psi)L(1 - \rho - \beta_3, \bar{\psi})}{L'(\rho, \psi)} (1 + O(\mathcal{L}^{-123})). \end{aligned} \quad (13.1)$$

By Lemma 6.1,

$$L(1 - \rho - \beta_3, \bar{\psi}) = \overline{L(\rho + \beta_3, \psi)} = K(1 - \rho - \beta_3, \bar{\psi}) + Z(\rho + \beta_3, \psi)^{-1} N(\rho + \beta_3, \psi) + O(E_1(\rho + \beta_3, \psi)),$$

and, by Lemma 5.1,

$$Z(\rho + \beta_3, \psi)^{-1} = Z(\rho + \beta_2, \psi)^{-1} (pt_0)^{\beta_1} + O(\mathcal{L}^{-123}).$$

Hence

$$\begin{aligned} &L(\rho + \beta_2, \psi)L(1 - \rho - \beta_3, \bar{\psi}) \\ &= L(\rho + \beta_2, \psi)K(1 - \rho - \beta_3, \bar{\psi}) + (pt_0)^{\beta_1} N(\rho + \beta_3, \psi)L(1 - \rho - \beta_2, \bar{\psi}) \\ &\quad + O(|L(\rho + \beta_2, \psi)N(\rho + \beta_3, \psi)\mathcal{L}^{-123}| + O(|L(\rho + \beta_2, \psi)|E_1(\rho + \beta_3, \psi))). \end{aligned} \quad (13.2)$$

By Lemma 6.1 and 5.1,

$$\begin{aligned} (pt_0)^{\beta_1} L(1 - \rho - \beta_2, \bar{\psi}) &= (pt_0)^{\beta_1} K(1 - \rho - \beta_2, \bar{\psi}) + (pt_0)^{\beta_3} Z(\rho, \psi)^{-1} N(\rho + \beta_2, \psi) \\ &\quad + O(N(\rho + \beta_2, \psi)\mathcal{L}^{-123}) + O(E_1(\rho + \beta_2, \psi)). \end{aligned}$$

We insert this into (13.2) and then insert the result into (13.1). Thus we obtain

$$\begin{aligned} \mathcal{C}^*(\rho, \psi) &= -i \frac{L(\rho + \beta_1, \psi)L(\rho + \beta_2, \psi)}{L'(\rho, \psi)} K(1 - \rho - \beta_3, \bar{\psi}) \\ &\quad - i(pt_0)^{\beta_1} \frac{L(\rho + \beta_1, \psi)}{L'(\rho, \psi)} N(\rho + \beta_3, \psi) K(1 - \rho - \beta_2, \bar{\psi}) \\ &\quad - i(pt_0)^{\beta_3} Z(\rho, \psi)^{-1} \frac{L(\rho + \beta_1, \psi)}{L'(\rho, \psi)} N(\rho + \beta_2, \psi) N(\rho + \beta_3, \psi) \\ &\quad + O(E_1^*(\rho, \psi)) \end{aligned} \quad (13.3)$$

where

$$\begin{aligned} E_1^*(\rho, \psi) &= \left| \frac{L(\rho + \beta_1, \psi)L(\rho + \beta_2, \psi)}{L'(\rho, \psi)} \right| (|L(\rho + \beta_3, \psi)|t_0^{-1} + |N(\rho + \beta_3, \psi)|\mathcal{L}^{-123} + E_1(\rho + \beta_3, \psi)) \\ &\quad + \left| \frac{L(\rho + \beta_1, \psi)N(\rho + \beta_3, \psi)}{L'(\rho, \psi)} \right| (|N(\rho + \beta_2, \psi)|\mathcal{L}^{-123} + E_1(\rho + \beta_2, \psi)). \end{aligned}$$

Recall that $B(s, \psi)$ is given by (12.2). Multiplying (13.3) by $Z(\rho, \chi\psi)^{-1}B(\rho, \psi)$ and applying Lemma 4.8 we obtain

$$\mathcal{C}^*(\rho, \psi)Z(\rho, \chi\psi)^{-1}B(\rho, \psi) = -i(\mathcal{K}_1^*(\rho, \psi) + (pt_0)^{\beta_1}\mathcal{K}_2^*(\rho, \psi) - (pt_0)^{\beta_3}\mathcal{K}_3^*(\rho, \psi)) + O(E_1^*(\rho, \psi)|B(\rho, \psi)|)$$

where

$$\mathcal{K}_1^*(\rho, \psi) = Z(\rho, \chi\psi)^{-1} \frac{L(\rho + \beta_1, \psi)L(\rho + \beta_2, \psi)}{L'(\rho, \psi)} B(\rho, \psi) K(1 - \rho - \beta_3, \bar{\psi}), \quad (13.4)$$

$$\mathcal{K}_2^*(\rho, \psi) = Z(\rho, \chi\psi)^{-1} \frac{L(\rho + \beta_1, \psi)}{L'(\rho, \psi)} B(\rho, \psi) N(\rho + \beta_3, \psi) K(1 - \rho - \beta_2, \bar{\psi}), \quad (13.5)$$

$$\mathcal{K}_3^*(\rho, \psi) = \frac{L(\rho + \beta_1, \psi)}{L'(\rho, \psi)} B(\rho, \psi) G(\rho, \psi) N(\rho + \beta_2, \psi) N(\rho + \beta_3, \psi) F(1 - \rho, \bar{\psi}), \quad (13.6)$$

Inserting this into (12.4) we obtain

$$\Xi_{14} = -i(\Phi_1 + \Phi_2 - \Phi_3) + O(\mathcal{E}) \quad (13.7)$$

where

$$\Phi_1 = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{K}_1^*(\rho, \psi) \omega(\rho), \quad (13.8)$$

$$\Phi_2 = \sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_1} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{K}_2^*(\rho, \psi) \omega(\rho), \quad (13.9)$$

$$\Phi_3 = \sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} \sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{K}_3^*(\rho, \psi) \omega(\rho), \quad (13.10)$$

and

$$\mathcal{E} = \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} E_1^*(\rho, \psi) |B(\rho, \psi)| \omega(\rho).$$

Combining (2.34), Cauchy's inequality, Proposition 7.1, Lemma 5.9, 6.1 and 3.3, we can verify that

$$\mathcal{E} = o(\mathcal{P}). \quad (13.11)$$

For example, by (2.34),

$$\begin{aligned} & \sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} \left| \frac{L(\rho + \beta_1, \psi)}{L'(\rho, \psi)} \right| |L(\rho + \beta_2, \psi)|^2 \omega(\rho) \\ &= -\frac{1}{2\pi} \sum_{\psi \in \Psi_1} \left(\int_{\mathcal{J}(\alpha)} - \int_{\mathcal{J}(-\alpha)} \right) \frac{M(s + \beta_1, \psi)}{M(s, \psi)} L(s + \beta_2, \psi) L(1 - s - \beta_2, \bar{\psi}) \omega(s) ds + O(\varepsilon), \end{aligned}$$

the right side being estimated via Lemma 5.9, 6.1 and 3.3.

14. Mean-value formula II

Recall that we always assume ψ is a primitive character (mod p), $p \sim P$. Sometimes we write p_ψ for the modulus p .

Let $\mathbf{k}^* = \{\kappa^*(m)\}$ and $\mathbf{a}^* = \{a^*(n)\}$ denote sequences of complex numbers satisfying

$$\kappa^*(m) \ll \tau_5(m), \quad (14.1)$$

$$a^*(n) \ll 1, \quad a^*(n) = 0 \quad \text{if } n > 2P_4. \quad (14.2)$$

For $\beta \in \mathbf{C}$ write

$$\Theta_2(\beta, \mathbf{k}^*, \mathbf{a}^*) = \sum_{\psi \in \Psi_1} \frac{(p_\psi t_0)^\beta}{2\pi i} \int_{\mathcal{J}(1)} Z(s, \chi\psi)^{-1} \left(\sum_m \frac{\kappa^*(m)\psi(m)}{m^s} \right) \left(\sum_n \frac{a^*(n)\bar{\psi}(n)}{n^{1-s}} \right) \omega(s) ds.$$

The goal of this section is to prove

Proposition 14.1. *Suppose $|\beta| < 5\alpha$. Then*

$$\Theta_2(\beta, \mathbf{k}^*, \mathbf{a}^*) = \frac{1}{\varphi(D)} \sum_{p \sim P} (pt_0)^\beta \sum_d \frac{1}{d} \sum_k \frac{\mu\chi(k)a^*(dk)}{k\varphi(k)} \sum_{(l,k)=1} \chi(l)\kappa^*(dl) \Delta\left(\frac{l}{Dpk}\right) + o(\mathcal{P}).$$

Proof. We prove this proposition with $\beta = 0$ only, as the general case is almost identical. Write Θ_2 for $\Theta_2(0, \mathbf{k}^*, \mathbf{a}^*)$. Similar to the proof of Proposition 7.1, in the expression for Θ_2 , we can extend the sum over Ψ_1 to the sum over Ψ , with acceptable errors. Namely we have

$$\Theta_2 = \sum_{\psi \in \Psi} \tilde{I}_2(\psi) + o(\mathcal{P}) \quad (14.3)$$

where

$$\tilde{I}_2(\psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(1)} Z(s, \chi\psi)^{-1} \left(\sum_m \frac{\kappa^*(m)\psi(m)}{m^s} \right) \left(\sum_n \frac{a^*(n)\bar{\psi}(n)}{n^{1-s}} \right) \omega(s) ds.$$

Assume $\psi \pmod{p} \in \Psi$. We use (2.5) with $\theta = \chi\psi$ and then replace the segment $\mathcal{J}(1)$ by the vertical line $\sigma = 3/2$ with a negligible error. Thus, by integration term by term,

$$\tilde{I}_2(\psi) = \frac{\tau(\chi\bar{\psi})}{Dp} \sum_m \sum_n \frac{\kappa^*(m)a^*(n)\psi(m)\bar{\psi}(n)}{n} \Delta_1\left(\frac{m}{Dpn}\right) + O(\varepsilon).$$

For $(mn, p) = 1$ we have

$$\sum_{\psi \pmod{p}} \tau(\chi\bar{\psi})\psi(m)\bar{\psi}(n) = \tau(\chi)\chi(p)(p-1)e\left(\frac{m\overline{Dn}}{p}\right).$$

Note that $(n, p) = 1$ if $n < 2P_4$ and $|\tau(\chi\psi_p^0)| = \sqrt{D}$. Hence, substituting $d = (m, n)$, $m = dl$, $n = dk$ and inverting the order of summation we obtain

$$\begin{aligned} \sum_{\psi \pmod{p}}^* \tilde{I}_2(\psi) &= \frac{\tau(\chi)\chi(p)}{D} \sum_m \sum_n \frac{\kappa^*(m)a^*(n)}{n} \Delta_1\left(\frac{m}{Dpn}\right) e\left(\frac{m\overline{Dn}}{p}\right) + O(PT^{-c}) \\ &= \frac{\tau(\chi)\chi(p)}{D} \sum_d \frac{1}{d} \sum_k \frac{a^*(dk)}{k} \sum_{(l,k)=1} \kappa^*(dl) \Delta_1\left(\frac{l}{Dpk}\right) e\left(\frac{l\overline{Dk}}{p}\right) + O(PT^{-c}). \end{aligned}$$

(here we have replaced the constraint $(l, pk) = 1$ by $(l, k) = 1$ with an acceptable error, and used a bound for $|\Delta_1(x)| = |\Delta(x)|$ given by Lemma 5.3). Since

$$\frac{\overline{Dk}}{p} \equiv -\frac{\bar{p}}{Dk} + \frac{1}{Dpk} \pmod{1},$$

it follows that

$$\Delta_1\left(\frac{l}{Dpk}\right) e\left(\frac{l\overline{Dk}}{p}\right) = \Delta\left(\frac{l}{Dpk}\right) e\left(-\frac{l\bar{p}}{Dk}\right)$$

with $\bar{p}p \equiv 1 \pmod{Dk}$. Hence

$$\begin{aligned} \sum_{\psi \pmod{p}}^* \tilde{I}_2(\psi) &= \frac{\tau(\chi)\chi(p)}{D} \sum_d \frac{1}{d} \sum_k \frac{a^*(dk)}{k} \sum_{(l,k)=1} \kappa^*(dl) \Delta\left(\frac{l}{Dpk}\right) e\left(-\frac{l\bar{p}}{Dk}\right) \\ &\quad + O(PT^{-c}). \end{aligned} \tag{14.4}$$

Assume $(k, p) = 1$. If $(l, k) = 1$ and $(l, D) = D_1$, then $(k, D_1) = 1$. Thus, substituting $(l, D) = D_1$ and $l = D_1 l_1$, we find that the innermost sum in (14.4) is equal to

$$\sum_{\substack{D=D_1 D_2 \\ (D_1, k)=1}} \sum_{(l_1, D_2 k)=1} (\kappa_1 * b)(D_1 dl_1) \Delta\left(\frac{l_1}{D_2 pk}\right) e\left(-\frac{l_1 \bar{p}}{D_2 k}\right).$$

Inserting this into (14.4) and rearranging the terms we conclude

$$\sum_{\psi \pmod{p}}^* \tilde{I}_2(\psi) = \frac{\tau(\chi)\chi(p)}{D} \sum_{D=D_1 D_2} \mathcal{S}(D_1, D_2; p) + O(PT^{-c}),$$

where

$$\mathcal{S}(D_1, D_2; p) = \sum_d \frac{1}{d} \sum_{(k, D_1)=1} \frac{a^*(dk)}{k} \sum_{(l, D_2 k)=1} \kappa^*(D_1 dl) \Delta\left(\frac{l}{D_2 pk}\right) e\left(-\frac{l\bar{p}}{D_2 k}\right)$$

(here we have rewritten l for l_1). Since $\chi(-1)\tau(\chi)^2 = D$, the proof of Proposition 14.1 is now reduced to showing that

$$\begin{aligned} & \sum_{p \sim P} \chi(p) \mathcal{S}(1, D; p) \\ &= \frac{\chi(-1)\tau(\chi)}{\varphi(D)} \sum_d \frac{1}{d} \sum_k \frac{\mu\chi(k)a^*(dk)}{k\varphi(k)} \sum_{(l,k)=1} \chi(l)\kappa^*(dl) \Delta\left(\frac{l}{Dpk}\right) \\ & \quad + O(P^2 D^{1/2-c}) \end{aligned} \quad (14.5)$$

and, for $D = D_1 D_2$, $D_1 > 1$,

$$\sum_{p \sim P} \chi(p) \mathcal{S}(D_1, D_2; p) \ll P^2 D^{1/2-c}. \quad (14.6)$$

We first prove (14.5). Assume $k < 2P_4$ and $(l, Dk) = 1$. Then

$$e\left(-\frac{l\bar{p}}{Dk}\right) = \frac{1}{\varphi(Dk)} \sum_{\theta \pmod{Dk}} \tau(\bar{\theta})\theta(-l)\bar{\theta}(p).$$

Hence

$$\mathcal{S}(1, D; p) = \sum_d \frac{1}{d} \sum_k \frac{a^*(dk)}{k\varphi(Dk)} \sum_{\theta \pmod{Dk}} \tau(\bar{\theta})\bar{\theta}(p) \sum_l \kappa^*(dl)\theta(-l) \Delta\left(\frac{l}{Dpk}\right) \quad (14.7)$$

(here we have remove the constraint $(l, Dk) = 1$ as it is superfluous).

If θ is the principal character ψ_{Dk}^0 , then $\tau(\bar{\theta}) = \mu(Dk)$. The total contribution from the $\theta = \psi_{Dk}^0$ term to the right side above is, by Lemma 5.3,

$$\ll \sum_d \frac{1}{d} \sum_k \sum_l \frac{|(\kappa_1 * b)(dl)a^*(dk)|}{\varphi(Dk)k} \left| \Delta\left(\frac{l}{Dpk}\right) \right| \ll P\mathcal{L}^c$$

which is admissible for (14.5). Now, let θ_k^1 be the character \pmod{Dk} induced by the primitive character $\chi \pmod{D}$ which is real. Then

$$\tau(\theta_k^1)\theta_k^1(p)\theta_k^1(-l) = \tau(\chi)\mu\chi(k)\chi(-pl).$$

Hence

$$\begin{aligned} & \sum_d \frac{1}{d} \sum_k \sum_{(l,Dk)=1} \frac{\kappa^*(dl)a^*(dk)}{\varphi(Dk)k} \Delta\left(\frac{l}{Dpk}\right) \tau(\theta_k^1)\theta_k^1(p)\theta_k^1(-l) \\ &= \frac{\chi(-p)\tau(\chi)}{\varphi(D)} \sum_d \frac{1}{d} \sum_k \frac{\mu\chi(k)a^*(dk)}{k\varphi(k)} \sum_{(l,k)=1} \chi(l)\kappa^*(dl) \Delta\left(\frac{l}{Dpk}\right). \end{aligned}$$

By the above discussion, to complete the proof of (14.5), it now suffices to show that the total contribution from the terms with $\theta \neq \psi_{Dk}^0$ and $\theta \neq \theta_k^1$ in (14.7) to the left side of (14.5) is $O(P^2D^{-c})$. Namely we need to prove

$$\sum_d \frac{1}{d} \sum_k \frac{|a^*(dk)|}{\varphi(Dk)k} \sum'_{\substack{\theta(\bmod Dk) \\ \theta \neq \theta_k^1}} |\tau(\bar{\theta})| \left| \sum_l \kappa^*(dl)\theta(l) \sum_{p \sim P} \chi^{\bar{\theta}(p)} \Delta\left(\frac{l}{Dpk}\right) \right| \quad (14.8)$$

$$\ll P^2D^{-c}.$$

If $\theta(\bmod Dk)$ is induced by a primitive character $\theta^*(\bmod r)$, then $r|Dk$ and $|\tau(\bar{\theta})| \leq \sqrt{r}$. For $r|Dk$ we have

$$\frac{Dk}{r} \equiv 0 \left(\bmod \frac{D}{(D, r)} \right).$$

Thus, substituting $Dk = hr$, we see that the left side of (14.8) is

$$\ll \sum_d \frac{1}{d} \sum_{1 < r < 2DP_4} \sum_{\substack{h < P/r \\ h \equiv 0(D/(D, r))}} \frac{D}{\varphi(hr)h\sqrt{r}} \sum'_{\substack{\theta(\bmod r) \\ \theta \neq \chi}} \left| \sum_{(l, h)=1} \kappa^*(dl)\theta(l) \sum_{p \sim P} \chi^{\bar{\theta}(p)} \Delta\left(\frac{l}{phr}\right) \right|.$$

The range for r is divided into two parts according to $1 < r < D^3$ and $D^3 \leq r < 2DP_4$. In a way similar to the proof of Proposition 7.1, for $1 < r < D^3$ we use the Mellin transform, Lemma 5.4 (i) and Lemma 5.6; for $D^3 \leq r < 2DP_4$ we use the Mellin transform, Lemma 5.4 (i) and the large sieve inequality. Thus we obtain (14.5).

The proof of (14.6) is analogous. In the case $D = D_1D_2$, $D_1 > 1$, the major difference from the proof of (14.5) is that the constraint $(k, D_1) = 1$ implies $D \nmid D_2k$, so that any non-principal character $\theta(\bmod D_2k)$ can not be induced by the real character $\chi(\bmod D)$. Thus, when the left side of (14.6) is treated in much the same way as (14.5), the main terms involved in the proof of (14.5) do not appear. This yields (14.6). \square

15. Evaluation of Φ_1

Recall that Φ_1 is given by (13.8). In view of (12.2), $B(s, \psi)$ can be written as

$$B(s, \psi) = \sum_n \frac{b(n)\chi\psi(n)}{n^s} \quad (15.1)$$

with

$$b(n) \ll \tau_2(n), \quad b(n) = 0 \quad \text{if } n > PT^{-2}\eta_+. \quad (15.2)$$

Write

$$\mathcal{K}_1(s, \psi) = Z(s, \chi\psi)^{-1} \frac{L(s + \beta_1, \psi)L(s + \beta_2, \psi)}{L(s, \psi)} B(s, \psi) K(1 - s - \beta_3, \bar{\psi}).$$

Similar to (8.1), for $\psi \in \Psi_1$ we have

$$\sum_{\rho \in \mathcal{Z}(\psi)} \mathcal{K}_1^*(\rho, \psi) \omega(\rho) = I_2^+(\psi) - I_2^-(\psi) + O(\varepsilon)$$

where

$$I_2^\pm(\psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(\pm\alpha)} \mathcal{K}_1(s, \psi) \omega(s) ds.$$

Hence

$$\Phi_1 = \sum_{\psi \in \Psi_1} (I_2^+(\psi) - I_2^-(\psi)) + O(\varepsilon). \quad (15.3)$$

First we prove that

$$\sum_{\psi \in \Psi_1} I_2^-(\psi) = o(\mathcal{P}). \quad (15.4)$$

Assume $\psi \in \Psi_1$ and $s \in \mathcal{J}(-\alpha)$. By (2.2) and Lemma 5.1,

$$\frac{L(s + \beta_1, \psi) L(s + \beta_2, \psi)}{L(s, \psi)} = (pt_0)^{-\beta_3} Z(s, \psi) \frac{L(1 - s - \beta_1, \bar{\psi}) L(1 - s - \beta_2, \bar{\psi})}{L(1 - s, \bar{\psi})} (1 + O(\mathcal{L}^{-123})).$$

In a way similar to the proof of Lemma 8.1, it can be shown that the total contribution of the $O(\mathcal{L}^{-123})$ term to the left side of (15.4) is $o(\mathcal{P})$. On the other hand, by (2.4), (2.5) and the relation

$$\tau(\chi\psi) = \tau(\chi)\tau(\psi)\psi(D)\chi(p),$$

we have

$$\frac{Z(s, \psi)}{Z(s, \chi\psi)} = \tau(\chi)\chi(p)\bar{\psi}(D)D^{s-1}(1 + O(e^{-\pi t})).$$

For $\sigma < 0$ we can write

$$\frac{L(1 - s - \beta_1, \bar{\psi}) L(1 - s - \beta_2, \bar{\psi}) K(1 - s - \beta_3, \bar{\psi})}{L(1 - s, \bar{\psi})} = \sum_m \frac{\tilde{k}(m) \bar{\psi}(Dm)}{m^{1-s}}$$

with $\tilde{k}(m) \ll \tau_4(m)$. Hence, moving the segment $\mathcal{J}(-\alpha)$ to $\mathcal{J}(-1)$ with a negligible error, we find that

$$\sum_{\psi \in \Psi_1} I_2^-(\psi) = \tau(\chi) \sum_{\psi \in \Psi_1} \chi(p_\psi) (p_\psi t_0)^{-\beta_3} \cdot \frac{1}{2\pi i} \int_{\mathcal{J}(-1)} \left(\sum_m \frac{\tilde{k}(m) \bar{\psi}(Dm)}{(Dm)^{1-s}} \right) B(s, \psi) \omega(s) ds + o(\mathcal{P}).$$

In a way similar to the proof of (7.3), the sum over $\psi \in \Psi_1$ can be extended to the sum over $\psi \in \Psi$, and then the segment $\mathcal{J}(-1)$ can be replaced by the line $\sigma = -1/2$, with acceptable errors. Thus the right side above is equal to

$$\tau(\chi) \sum_{p \sim P} \chi(p) (pt_0)^{-\beta_3} \sum_{\psi(\bmod p)}^* \frac{1}{2\pi i} \int_{(-1/2)} \left(\sum_m \frac{\tilde{k}(m) \bar{\psi}(Dm)}{(Dm)^{1-s}} \right) B(s, \psi) \omega(s) ds + o(\mathcal{P}).$$

Since

$$\frac{1}{2\pi i} \int_{(-1/2)} \frac{\omega(s) ds}{n^s (Dm)^{1-s}} = \frac{1}{Dm} \left(\frac{Dm}{n} \right)^{s_0} \exp \left\{ -\mathcal{L}_2^2 \log^2 \frac{Dm}{n} \right\}$$

and, for $n < PT^{-5}$ and $Dm < 2n$,

$$\sum_{\psi \pmod{p}}^* \psi(n) \bar{\psi}(Dm) \ll \begin{cases} p & \text{if } n = Dm, \\ 1 & \text{if } n \neq Dm, \end{cases}$$

it follows that

$$\sum_{\psi \pmod{p}}^* \frac{1}{2\pi i} \int_{(-1/2)} \left(\sum_m \frac{\tilde{k}(m) \bar{\psi}(Dm)}{(Dm)^{1-s}} \right) B(s, \psi) \omega(s) ds \ll PD^{-4/5}.$$

This yields (15.4).

Let $\kappa_1(m)$ be given by

$$\frac{\zeta(s + \beta_1) \zeta(s + \beta_2)}{\zeta(s)} = \sum_m \frac{\kappa_1(m)}{m^s}, \quad \sigma > 1.$$

Regarding b as an arithmetic function, for $\sigma > 1$ we have

$$\frac{L(s + \beta_1, \psi) L(s + \beta_2, \psi) B(s, \psi)}{L(s, \psi)} = \sum_m \frac{(\kappa_1 * b)(m) \psi(m)}{m^s}.$$

On the other hand, we can write

$$K(1 - s - \beta_3, \bar{\psi}) = \sum_n \frac{\tilde{g}_3(n) \bar{\psi}(n)}{n^{1-s}}$$

with

$$\tilde{g}_3(y) = y^{\beta_3} g^*(P_4/y).$$

Hence, moving the segment $\mathcal{J}(\alpha)$ to $\mathcal{J}(1)$ with a negligible error, we obtain

$$\sum_{\psi \in \Psi_1} I_2^+(\psi) = \Theta_2(0, \mathbf{k}_1^*, \mathbf{a}_1^*) + O(\varepsilon) \tag{15.5}$$

with

$$\kappa_1^*(m) = (\kappa_1 * b)(m), \quad a_1^*(n) = \tilde{g}_3(n).$$

It follows by (15.3)-(15.5) and Proposition 14.1 that

$$\Phi_1 = \sum_{p \sim P} \Phi_1(p) + o(\mathcal{P}) \tag{15.6}$$

where

$$\Phi_1(p) = \frac{1}{\varphi(D)} \sum_k \frac{\mu\chi(k)}{k\varphi(k)} \sum_d \frac{\tilde{g}_3(dk)}{d} \sum_{(l,k)=1} (\kappa_1 * b)(dl) \chi(l) \Delta\left(\frac{l}{Dpk}\right). \quad (15.7)$$

Assume $dk < 2P_4$. Similar to (7.17),

$$(\kappa_1 * b)(dl) = \sum_{d=d_1d_2} \sum_{\substack{l=l_1l_2 \\ (l_1,d_2)=1}} b(d_2l_2) \kappa_1(d_1l_1).$$

This yields

$$\sum_{(l,k)=1} (\kappa_1 * b)(dl) \chi(l) \Delta\left(\frac{l}{Dpk}\right) = \sum_{d=d_1d_2} \sum_{(l_2,k)=1} b(d_2l_2) \chi(l_2) \sum_{(l_1,d_2k)=1} \kappa_1(d_1l_1) \chi(l_1) \Delta\left(\frac{l_1l_2}{Dpk}\right).$$

The innermost sum above is, by the Mellin transform, equal to

$$\frac{1}{2\pi i} \int_{(2)} \left(\sum_{(l_1,d_2k)=1} \frac{\kappa_1(d_1l_1) \chi(l_1)}{l_1^s} \right) \left(\frac{Dpk}{l_2} \right)^s \delta(s) ds. \quad (15.8)$$

Every l_1 can be uniquely written as $l_1 = hl$ such that $h \in \mathcal{N}(d_1)$ and $(l, d_1) = 1$, so that $\kappa_1(d_1l_1) = \kappa_1(d_1h) \kappa_1(l)$. Hence, for $\sigma > 1$,

$$\begin{aligned} \sum_{(l_1,d_2k)=1} \frac{\kappa_1(d_1l_1) \chi(l_1)}{l_1^s} &= \tilde{\kappa}_1(d_1; d_2k, s) \sum_{(l,d_1d_2k)=1} \frac{\kappa_1(l) \chi(l)}{l^s} \\ &= \tilde{\kappa}_1(d_1; d_2k, s) \lambda_1(d_1d_2k, s) \frac{L(s + \beta_1, \chi) L(s + \beta_2, \chi)}{L(s, \chi)} \end{aligned}$$

where

$$\tilde{\kappa}_1(d_1; r, s) = \sum_{\substack{h \in \mathcal{N}(d_1) \\ (h,r)=1}} \frac{\kappa_1(d_1h) \chi(h)}{h^s} \quad (15.9)$$

and

$$\lambda_1(n, s) = \prod_{q|n} \frac{(1 - \chi(q)q^{-s-\beta_1})(1 - \chi(q)q^{-s-\beta_2})}{1 - \chi(q)q^{-s}}. \quad (15.10)$$

For notational simplicity we shall write $\tilde{\kappa}_1(d; r)$ and $\lambda_1(n)$ for $\tilde{\kappa}_1(d; r, 1)$ and $\lambda_1(n, 1)$ respectively. Note that $Dpk/l_2 > T$ if $l_2 < PT^{-2}$. In a way similar to the treatment of (7.19), by Lemma 5.5, we see that the expression (15.8) is equal to the residue of the integrand at $s = \tilde{\rho}$ plus an acceptable error. Further, by (5.15), in the expression for this residue, we can replace $\tilde{\rho}$ by 1 with an acceptable error. Thus we have

$$\sum_{(l_1,d_2k)=1} \kappa_1(d_1l_1) \chi(l_1) \Delta\left(\frac{l_1l_2}{Dpk}\right) = \mathcal{R}_1^* \left(\frac{Dpk}{l_2} \right) \tilde{\kappa}_1(d_1; d_2k) \lambda_1(d_1d_2k) + O\left(\frac{\alpha^{100} \tau_3(d_1) Dpk}{l_2}\right),$$

where

$$\mathcal{R}_1^* = \frac{L(1 + \beta_1, \chi)L(1 + \beta_2, \chi)}{L'(1, \chi)} \delta(1).$$

This yields

$$\begin{aligned} \sum_{(l,k)=1} (\kappa_1 * b)(dl) \chi(l) \Delta\left(\frac{l}{Dpk}\right) &= \mathcal{R}_1^* Dpk \sum_{d=d_1 d_2} \tilde{\kappa}_1(d_1; d_2 k) \lambda_1(dk) \sum_{(l_2, k)=1} \frac{b(d_2 l_2) \chi(l_2)}{l_2} \\ &\quad + O(\alpha^{50} \tau_3(d_1) Dpk). \end{aligned}$$

Hence

$$\begin{aligned} \sum_d \frac{\tilde{g}_3(dk)}{d} \sum_{(l,k)=1} (\kappa_1 * b)(dl) \chi(l) \Delta\left(\frac{l}{Dpk}\right) \\ = \mathcal{R}_1^* Dpk \sum_{d_1} \sum_{d_2} \frac{\tilde{g}_3(d_1 d_2 k)}{d_1 d_2} \tilde{\kappa}_1(d_1; d_2 k) \lambda_1(d_1 d_2 k) \sum_{(l_2, k)=1} \frac{b(d_2 l_2) \chi(l_2)}{l_2} \\ + O(\alpha^{30} Dpk) \end{aligned}$$

Inserting this into (15.7) and rewriting d , l and m for d_2 , l_2 and d_1 respectively, we obtain

$$\Phi_1(p) = \frac{\mathcal{R}_1^* Dp}{\varphi(D)} \sum_d \sum_l \frac{b(dl) \chi(l)}{dl} \mathcal{D}_1(d, l) + o(P) \quad (15.11)$$

where

$$\mathcal{D}_1(d, l) = \sum_{(k,l)=1} \frac{\mu \chi(k)}{\varphi(k)} \sum_m \frac{\tilde{\kappa}_1(m; dk) \lambda_1(mdk) \tilde{g}_3(mdk)}{m}.$$

On substituting $n = mk$ we can write

$$\mathcal{D}_1(d, l) = \sum_n \frac{\lambda_1(dn) \tilde{g}_3(dn)}{n} \xi_1(n; d, l) \quad (15.12)$$

with

$$\xi_1(n; d, l) = \sum_{\substack{n=mk \\ (k,l)=1}} \frac{\mu \chi(k) k}{\varphi(k)} \tilde{\kappa}_1(m; dk). \quad (15.13)$$

It can be verified, for given d and l , that $\xi_1(n; d, l)$ is a multiplicative function of n . On the other hand, we have

$$\lambda_1(dn) = \lambda_1(d) \tilde{\lambda}_1(n, d)$$

with

$$\tilde{\lambda}_1(n, d) = \prod_{\substack{q|n \\ (q,d)=1}} \lambda_1(q).$$

Hence

$$\mathcal{D}_1(d, l) = \lambda_1(d) \sum_n \frac{\tilde{\lambda}_1(n, d) \xi_1(n; d, l) \tilde{g}_3(dn)}{n}.$$

On the right side above, we can replace the factor $\tilde{g}_3(dn)$ by $(dn)^{\beta_3} g(P_4/(dn))$ with a negligible error. Since

$$y^{\beta_3} g\left(\frac{P_4}{y}\right) = \frac{1}{2\pi i} \int_{(1)} \frac{P_4^{s+\beta_3}}{y^s} \frac{\omega_1(s + \beta_3)}{s + \beta_3} ds,$$

it follows that

$$\mathcal{D}_1(d, l) = \lambda_1(d) \cdot \frac{1}{2\pi i} \int_{(1)} \left(\sum_n \frac{\tilde{\lambda}_1(n, d) \xi_1(n; d, l)}{n^{1+s}} \right) \frac{P_4^{s+\beta_3}}{d^s} \frac{\omega_1(s + \beta_3)}{s + \beta_3} ds + O(\varepsilon). \quad (15.14)$$

Note that $\tilde{\lambda}_1(q^r, d) = \tilde{\lambda}_1(q, d)$ for any r . If $\sigma > 1$, then

$$\sum_n \frac{\tilde{\lambda}_1(n, d) \xi_1(n; d, l)}{n^s} = \prod_q \left(1 + \tilde{\lambda}_1(q, d) \sum_r \frac{\xi_1(q^r; d, l)}{q^{rs}} \right).$$

If $(q, dl) = 1$, then

$$\xi_1(q; d, l) = \sum_{h \in \mathcal{N}(q)} \frac{\kappa_1(qh) \chi(h)}{h} - \frac{\chi(q)q}{q-1} = q^{-\beta_1} + q^{-\beta_2} - 1 - \chi(q) + O(1/q),$$

so that

$$\frac{(1 - q^{-s-\beta_1})(1 - q^{-s-\beta_2})}{(1 - q^{-s})(1 - \chi(q)q^{-s})} \left(1 + \tilde{\lambda}_1(q, d) \sum_r \frac{\xi_1(q^r; d, l)}{q^{rs}} \right) = 1 + O(q^{-19/10})$$

for $\sigma > 9/10$. In case $(q, dl) > 1$ and $\sigma > 9/10$, the left side above is trivially

$$1 + O(q^{-9/10}).$$

It follows that the function

$$\mathcal{M}_1(d, l; s) := \frac{\zeta(s)L(s, \chi)}{\zeta(s + \beta_1)\zeta(s + \beta_2)} \sum_n \frac{\tilde{\lambda}_1(n, d) \xi_1(n; d, l)}{n^s}$$

is analytic and it satisfies

$$\mathcal{M}_1(d, l; s) \ll \prod_{q|dl} \left(1 + \frac{c}{q^{9/10}} \right)$$

for $\sigma > 9/10$. The right side of (15.14) can be rewritten as

$$\lambda_1(d) \cdot \frac{1}{2\pi i} \int_{(1)} \frac{\zeta(1+s+\beta_1)\zeta(1+s+\beta_2)\mathcal{M}_1(d, l; 1+s)}{\zeta(1+s)L(1+s, \chi)} \frac{P_4^{s+\beta_3}}{d^s} \frac{\omega_1(s+\beta_3)}{s+\beta_3} ds + O(\varepsilon).$$

Assume $dl < PT^{-2}$, and $(dl, D) = 1$. Note that $P_4/d > T$. In a way similar to the proof of Lemma 8.4, we deduce that

$$\mathcal{D}_1(d, l) = \lambda_1(d) \sum_{j \leq 3} \mathcal{R}_{1j} d^{\beta_j} \mathcal{M}_1(d, l; 1 - \beta_j) + O(\varepsilon_1) \quad (15.15)$$

where \mathcal{R}_{1j} , $1 \leq j \leq 3$, is the residue of the function

$$\frac{\zeta(1+s+\beta_1)\zeta(1+s+\beta_2)}{\zeta(1+s)L(1+s, \chi)} \frac{P_4^{s+\beta_3}\omega_1(s+\beta_3)}{s+\beta_3} \quad (15.16)$$

at $s = -\beta_j$ (the function (15.16) also has a simple pole at $s = \tilde{\rho} - 1$, while the residue at this point can be regarded as an acceptable error). Inserting this into (15.11) and substituting $n = dl$ we obtain

$$\Phi_1(p) = \frac{\mathcal{R}_1^* D p}{\varphi(D)} \sum_{j \leq 3} \mathcal{R}_{1j} \mathcal{S}_{1j} + o(p) \quad (15.17)$$

where

$$\mathcal{S}_{1j} = \sum_n \frac{b(n)}{n} \sum_{n=dl} \lambda_1(d) d^{\beta_j} \chi(l) \mathcal{M}_1(d, l; 1 - \beta_j).$$

If $(q, dl) = 1$, then $\tilde{\lambda}_1(q, d) = \tilde{\lambda}_1(q, 1)$ and $\xi_1(q^r; d, l) = \xi_1(q^r; 1, 1)$ for any r . Hence

$$\frac{\mathcal{M}_1(d, l; s)}{\mathcal{M}_1(1, 1; s)} = \prod_{q|dl} \left(1 + \tilde{\lambda}_1(q, d) \sum_r \frac{\xi_1(q^r; d, l)}{q^{rs}} \right) \left(1 + \tilde{\lambda}_1(q, 1) \sum_r \frac{\xi_1(q^r; 1, 1)}{q^{rs}} \right)^{-1}. \quad (15.18)$$

We can rewrite

$$\mathcal{S}_{1j} = \mathcal{M}_1(1, 1; 1 - \beta_j) \sum_n \frac{b(n) \varpi_{1j}(n)}{n} \quad (15.19)$$

with

$$\varpi_{1j}(n) = \sum_{n=dl} \lambda_1(d) d^{\beta_j} \chi(l) \frac{\mathcal{M}_1(d, l; 1 - \beta_j)}{\mathcal{M}_1(1, 1; 1 - \beta_j)}.$$

In view of (15.18), we see that $\varpi_{1j}(n)$ is a multiplicative function.

Let

$$\mathcal{Q} = \prod_{q < D^4} q.$$

Every n can be uniquely written as $n = n_1 n_2$ with $n_1 \in \mathcal{N}(\mathcal{Q})$ and $(n_2, \mathcal{Q}) = 1$. Hence

$$\sum_n \frac{b(n)\varpi_{1j}(n)}{n} = \sum_{n_1 \in \mathcal{N}(\mathcal{Q})} \frac{\varpi_{1j}(n_1)}{n_1} \sum_{(n, \mathcal{Q})=1} \frac{b(n_1 n)\varpi_{1j}(n)}{n}. \quad (15.20)$$

Using the Rankin trick (see [14, Section 13.2], for example), we can impose the constraint $n_1 < T$ to the right side with an acceptable error $O(\varepsilon_1)$. In addition, by (15.18), for $n < PT^{-3}$, $(n, \mathcal{Q}) = 1$ and $dl = n$,

$$\lambda_1(d) \frac{\mathcal{M}_1(d, l; s)}{\mathcal{M}_1(1, 1; s)} = 1 + O(D^{-c}),$$

so that

$$\varpi_{1j}(n) = \chi(n)\varrho_j^*(n) + O(\tau_2(n)D^{-c}) \quad (15.21)$$

where

$$\varrho_j^*(n) = \sum_{d|n} d^{\beta_j} \chi(d).$$

The following lemma will be proved in Appendix B.

Lemma 15.1. *Suppose $n_1 \in \mathcal{N}(\mathcal{Q})$ and $n_1 < T$. Then*

$$\sum_{(n, \mathcal{Q})=1} \frac{b(n_1 n)\chi(n)\varrho_j^*(n)}{n} = \chi(n_1)\tau_2(n_1)\mathbf{e}_j + O(\alpha_1\tau_2(n_1))$$

where

$$\mathbf{e}_j = (e_{1j} + \iota_2 e_{2j})(\bar{\iota}_3 e_{3j} + \bar{\iota}_4 e_{2j})$$

with

$$e_{2j} = \left(1 - \frac{2j}{5} + \frac{8j}{25\pi i}\right) \exp\{5\pi i/4\} - \frac{8j}{25\pi i},$$

$$e_{3j} = \left(1 - \frac{2j}{3} + \frac{j}{1.1205\pi i}\right) \exp\{0.747\pi i\} - \frac{j}{1.1205\pi i}$$

and

$$e_{1j} = e'_{1j} - e''_{1j},$$

$$e'_{1j} = \left(1 - \frac{2j}{3} + \frac{j}{1.134\pi i}\right) \exp\{0.756\pi i\} - \frac{j}{1.134\pi i},$$

$$e''_{1j} = \frac{j}{0.756} \int_0^{0.004} (\exp\{(3/2)(0.504 - z)\pi i\} - \exp\{(3/4)\pi i\}) dz.$$

By (15.19)-(15.21) and Lemma 15.1 we obtain

$$\mathcal{S}_{1j} = \epsilon_j \mathcal{M}_1(1, 1; 1 - \beta_j) \sum_{\substack{n \in \mathcal{N}(\mathcal{Q}) \\ n < T}} \frac{\chi(n) \tau_2(n) \varpi_{1j}(n)}{n} + O(1/\mathcal{L}). \quad (15.22)$$

This remains valid if the constraint $n \in \mathcal{N}(\mathcal{Q})$ on the right side is removed, since, for $D^4 < q < T$,

$$\varrho_j^*(q) = \varrho_j(q) + O(\nu(q)) \ll \alpha_1 + \nu(q).$$

This yields, by (15.21),

$$\sum_{\substack{n \notin \mathcal{N}(\mathcal{Q}) \\ n < T}} \frac{\chi(n) \tau_2(n) \varpi_{1j}(n)}{n} \ll 1/\mathcal{L}.$$

To apply (15.22) we need two lemmas which will be proved in Appendix A.

Lemma 15.2. *If $|s - 1| < 5\alpha$, then*

$$\mathcal{M}_1(1, 1; s) = \prod_{(q,D)=1} \frac{1 - \chi(q)q^{-2}}{1 - q^{-2}} + O(\alpha_1).$$

Lemma 15.3. *For $\sigma \geq 9/10$ the function*

$$\mathcal{U}_{1j}(s) := \frac{1}{\zeta(s)^2 L(s, \chi)^2} \sum_n \frac{\chi(n) \tau_2(n) \varpi_{1j}(n)}{n^s}$$

is analytic and bounded. Further we have

$$\mathcal{U}_{2j}(1) = \frac{\varphi(D)^2}{D^2} \prod_{(q,D)=1} \frac{(1 - q^{-2})^2}{1 - \chi(q)q^{-2}} + O(\alpha_1).$$

By (4.2) and (4.3),

$$\begin{aligned} \sum_{n < T} \frac{\chi(n) \tau_2(n) \varpi_{1j}(n)}{n} &= \sum_n \frac{\chi(n) \tau_2(n) \varpi_{1j}(n)}{n} g\left(\frac{T}{n}\right) + O(\alpha_1) \\ &= \frac{1}{2\pi i} \int_{(1)} \zeta(1+s)^2 L(1+s, \chi)^2 \mathcal{U}_{2j}(1+s) \frac{T^s \omega_1(s) ds}{s} + O(\alpha_1). \end{aligned}$$

By Lemma 15.3, we can move the contour of integration in the same way as in the proof of Lemma 8.4 to obtain

$$\sum_{n < T} \frac{\chi(n) \tau_2(n) \varpi_{1j}(n)}{n} = L'(1, \chi)^2 \mathcal{U}_{2j}(1) + O(\alpha_1).$$

This together with Lemma 15.2 and 15.3 yields

$$\mathcal{M}_1(1, 1; 1 - \beta_j) \sum_{\substack{n \in \mathcal{N}(\mathcal{Q}) \\ n < T}} \frac{\chi(n) \tau_2(n) \overline{\omega}_{1j}(n)}{n} = \mathfrak{a} \frac{\varphi(D)}{D} + O(1/\mathcal{L}^3),$$

since

$$\frac{\varphi(D)}{D} \prod_{(q,D)=1} (1 - q^{-2}) = \frac{6}{\pi^2} \prod_{q|D} \frac{q}{q+1}.$$

It follows by (15.22) that

$$\mathcal{S}_{1j} = \mathfrak{e}_j \mathfrak{a} \frac{\varphi(D)}{D} + O(1/\mathcal{L}^3) \quad (15.23)$$

By Lemma 5.8,

$$\mathcal{R}_1^* = \beta_1 \beta_2 L'(1, \chi) + O(1/\mathcal{L}^{24})$$

and

$$\mathcal{R}_{1j} = \frac{P_4^{\beta_3 - \beta_j}}{(\beta_{j+1} - \beta_j)(\beta_{j+2} - \beta_j) L'(1, \chi)} + O(1/\mathcal{L}^3).$$

Hence, by direct calculation,

$$\mathcal{R}_1^* \mathcal{R}_{11} = 1 + O(1/\mathcal{L}),$$

$$\mathcal{R}_1^* \mathcal{R}_{12} = 2 + O(1/\mathcal{L}),$$

$$\mathcal{R}_1^* \mathcal{R}_{13} = 1 + O(1/\mathcal{L}).$$

Combining these relations with (15.23), (15.17) and (15.6) we conclude

$$\Phi_1 = (\mathfrak{e}_1 + 2\mathfrak{e}_2 + \mathfrak{e}_3) \mathfrak{a} \mathcal{P} + o(\mathcal{P}) \quad (15.24)$$

16. Evaluation of Φ_2

Recall that Φ_2 is given by (13.9). Write

$$\mathcal{K}_2(s, \psi) = Z(s, \chi\psi)^{-1} \frac{L(s + \beta_1, \psi)}{L(s, \psi)} B(s, \psi) N(s + \beta_3, \psi) K(1 - s - \beta_2, \bar{\psi}).$$

Similar to (15.3),

$$\Phi_2 = \sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_1} (I_3^+(\psi) - I_3^-(\psi)) + O(\varepsilon)$$

where

$$I_3^\pm(\psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(\pm\alpha)} \mathcal{K}_2(s, \psi) \omega(s) ds.$$

Note that the length of the sum $B(s, \psi)N(s + \beta_3, \psi)$ is $\leq PT^{-1}$. For $\psi \in \Psi_1$, moving the segment $\mathcal{J}(-\alpha)$ to $\mathcal{J}(-\mathcal{L}^9)$ we obtain, by simple estimation,

$$I_3^-(\psi) \ll \varepsilon.$$

Hence

$$\Phi_2 = \sum_{\psi \in \Psi_1} (pt_0)^{\beta_1} I_3^+(\psi) + O(\varepsilon).$$

In a way similar to the proof of (15.6), we can move the segment $\mathcal{J}(\alpha)$ to $\mathcal{J}(1)$ and then extend the sum over Ψ_1 to that over Ψ with an acceptable error. Hence

$$\Phi_2 = \sum_{p \sim P} (pt_0)^{\beta_1} \sum_{\psi(\bmod p)}^* \frac{1}{2\pi i} \int_{\mathcal{J}(1)} \mathcal{K}_2(s, \psi) \omega(s) ds + o(\mathcal{P}). \quad (16.1)$$

Let $\kappa_2(n)$ and $b_1(n)$ be given by

$$\begin{aligned} \sum_n \frac{\kappa_2(n)}{n^s} &= \frac{\zeta(s + \beta_1)}{\zeta(s)}, \quad \sigma > 1, \\ \sum_n \frac{b_1(n)\psi(n)}{n^s} &= B(s, \psi)N(s + \beta_3, \psi). \end{aligned}$$

Note that

$$K(1 - s - \beta_2, \bar{\psi}) = \sum_n \frac{\tilde{g}_2(n)\bar{\psi}(n)}{n^{1-s}}$$

with

$$\tilde{g}_2(y) = y^{\beta_2} g^*(P_4/y).$$

We can rewrite (16.1) as

$$\Phi_2 = \Theta_2(\beta_1, \mathbf{k}_2^*, \mathbf{a}_2^*) + o(p)$$

with $\kappa_2^* = \kappa_2 * b_1$ and $a_2^* = \tilde{g}_2$. It follows by Proposition 14.1 that

$$\Phi_2 = \sum_{p \sim P} (pt_0)^{\beta_1} \Phi_2(p) + o(\mathcal{P}) \quad (16.2)$$

where

$$\Phi_2(p) = \frac{1}{\varphi(D)} \sum_k \frac{\mu\chi(k)}{k\varphi(k)} \sum_d \frac{\tilde{g}_2(dk)}{d} \sum_{(l,k)=1} (\kappa_2 * b_1)(dl) \chi(l) \Delta\left(\frac{l}{Dpk}\right). \quad (16.3)$$

Similar to (7.17),

$$\begin{aligned} & \sum_{(l,k)=1} (\kappa_2 * b_1)(dl) \chi(l) \Delta\left(\frac{l}{Dpk}\right) \\ &= \sum_{d=d_1 d_2} \sum_{(l_2, k)=1} b_1(d_2 l_2) \chi(l_2) \sum_{(l_1, d_2 k)=1} \kappa_2(d_1 l_1) \chi(l_1) \Delta\left(\frac{l_1 l_2}{Dpk}\right). \end{aligned}$$

The innermost sum is, by the Mellin transform, equal to

$$\frac{1}{2\pi i} \int_{(2)} \left(\sum_{(l_1, d_2 k)=1} \frac{\kappa_2(d_1 l_1) \chi(l_1)}{l_1^s} \right) \left(\frac{Dpk}{l_2} \right)^s \delta(s) ds. \quad (16.4)$$

For $\sigma > 1$,

$$\sum_{(l_1, d_2 k)=1} \frac{\kappa_2(d_1 l_1) \chi(l_1)}{l_1^s} = \tilde{\kappa}_2(d_1; d_2 k, s) \lambda_2(d_1 d_2 k, s) \frac{L(s + \beta_1, \chi)}{L(s, \chi)}$$

where

$$\tilde{\kappa}_2(d_1; r, s) = \sum_{\substack{h \in \mathcal{N}(d_1) \\ (h, r)=1}} \frac{\kappa_2(d_1 h) \chi(h)}{h^s}$$

and

$$\lambda_2(n, s) = \prod_{q|n} \frac{1 - \chi(q) q^{-s - \beta_1}}{1 - \chi(q) q^{-s}}.$$

For notational simplicity we shall write $\tilde{\kappa}_2(m; r)$ and $\lambda_2(n)$ for $\tilde{\kappa}_2(m; r, 1)$ and $\lambda_2(n, 1)$ respectively. Note that $Dpk/l_2 > T$ if $l_2 < P_2^2$. In a way similar to the treatment of (7.19), we see that the expression (16.4) is equal to the residue of the integrand at $s = \tilde{\rho}$ plus an acceptable error. Further, by (5.15), we can replace $\tilde{\rho}$ by 1 in the expression for this residue, with an acceptable error. Thus we have

$$\sum_{(l_1, d_2 k)=1} \kappa_2(d_1 l_1) \chi(l_1) \Delta \left(\frac{l_1 l_2}{Dpk} \right) = \mathcal{R}_2^* \left(\frac{Dpk}{l_2} \right) \tilde{\kappa}_2(d_1; d_2 k) \lambda_2(d_1 d_2 k) + O \left(\frac{\alpha^{100} \tau_2(d_1) Dpk}{l_2} \right)$$

where

$$\mathcal{R}_2^* = \frac{L(1 + \beta_1, \chi)}{L'(1, \chi)} \delta(1).$$

This yields

$$\begin{aligned} \sum_{(l, k)=1} (\kappa_2 * b_1)(dl) \chi(l) \Delta \left(\frac{l}{Dpk} \right) &= \mathcal{R}_2^* Dpk \sum_{d=d_1 d_2} \tilde{\kappa}_2(d_1; d_2 k) \lambda_2(dk) \sum_{(l_2, k)=1} \frac{b_1(d_2 l_2) \chi(l_2)}{l_2} \\ &\quad + O(\alpha^{50} \tau_3(d) Dpk). \end{aligned}$$

Hence

$$\begin{aligned} \sum_d \frac{\tilde{g}_2(dk)}{d} \sum_{(l, k)=1} (\kappa_2 * b_1)(dl) \chi(l) \Delta \left(\frac{l}{Dpk} \right) \\ = \mathcal{R}_2^* Dpk \sum_{d_1} \sum_{d_2} \frac{\tilde{g}_2(d_1 d_2 k)}{d_1 d_2} \tilde{\kappa}_2(d_1; d_2 k) \lambda_2(d_1 d_2 k) \sum_{(l_2, k)=1} \frac{b_1(d_2 l_2) \chi(l_2)}{l_2} \\ + O(\alpha^{20} Dpk). \end{aligned}$$

Inserting this into (16.3) and rewriting d, l and m for d_2, l_2 and d_1 respectively, we obtain

$$\Phi_2(p) = \frac{\mathcal{R}_2^* D p}{\varphi(D)} \sum_d \sum_l \frac{b_1(dl)\chi(l)}{dl} \mathcal{D}_2(d, l) + o(1) \quad (16.5)$$

where

$$\mathcal{D}_2(d, l) = \sum_{(k,l)=1} \frac{\mu\chi(k)}{\varphi(k)} \sum_m \frac{\tilde{\kappa}_2(m; dk)\lambda_2(mdk)\tilde{g}_2(mdk)}{m}.$$

On substituting $n = mk$ we can write

$$\mathcal{D}_2(d, l) = \sum_n \frac{\lambda_2(dn)\tilde{g}_2(dn)}{n} \xi_2(n; d, l) \quad (16.6)$$

with

$$\xi_2(n; d, l) = \sum_{\substack{n=mk \\ (k,l)=1}} \frac{\mu\chi(k)k}{\varphi(k)} \tilde{\kappa}_2(m; dk). \quad (16.7)$$

It can be verified, for given d and l , that $\xi_2(n; d, l)$ is a multiplicative function of n , and

$$\mathcal{D}_2(d, l) = \lambda_2(d) \sum_n \frac{\tilde{\lambda}_2(n, d)\xi_2(n; d, l)\tilde{g}_2(dn)}{n}$$

with

$$\tilde{\lambda}_2(n, d) = \prod_{\substack{q|n \\ (q,d)=1}} \lambda_2(q). \quad (16.8)$$

Hence, similar to (15.14),

$$\mathcal{D}_2(d, l) = \lambda_2(d) \cdot \frac{1}{2\pi i} \int_{(1)} \left(\sum_n \frac{\tilde{\lambda}_2(n, d)\xi_2(n; d, l)}{n^{1+s}} \right) \frac{P_4^{s+\beta_2}}{d^s} \frac{\omega_1(s + \beta_2)}{s + \beta_2} ds + O(\varepsilon). \quad (16.9)$$

It can be verified, for $\sigma > 9/10$, that the function

$$\mathcal{M}_2(d, l; s) := \frac{\zeta(s)L(s, \chi)}{\zeta(s + \beta_1)} \sum_n \frac{\tilde{\lambda}_2(n, d)\xi_2(n; d, l)}{n^s}$$

is analytic and it satisfies

$$\mathcal{M}_2(d, l; s) \ll \prod_{q|dl} \left(1 + \frac{c}{q^{9/10}} \right).$$

Thus we can rewrite $\mathcal{D}_2(d, l)$ as

$$\lambda_2(d) \cdot \frac{1}{2\pi i} \int_{(1)} \frac{\zeta(1+s+\beta_1) \mathcal{M}_2(d, l; 1+s) P_4^{s+\beta_2}}{\zeta(1+s)L(1+s, \chi)} \frac{\omega_1(s+\beta_2)}{d^s} \frac{ds}{s+\beta_2} + O(\varepsilon).$$

Assume $dl < P_2^2$, and $(dl, D) = 1$. Note that $P_4/d > T$. In a way similar to the proof of (15.15), we deduce that

$$\mathcal{D}_2(d, l) = \lambda_2(d) \sum_{j=1,2} \mathcal{R}_{2j} d^{\beta_j} \mathcal{M}_2(d, l; 1-\beta_j) + O(\varepsilon_1) \quad (16.10)$$

where \mathcal{R}_{2j} , $j = 1, 2$, is the residue of the function

$$\frac{\zeta(1+s+\beta_1)}{\zeta(1+s)L(1+s, \chi)} \frac{P_4^{s+\beta_2} \omega_1(s+\beta_2)}{s+\beta_2} \quad (16.11)$$

at $s = -\beta_j$. Inserting this into (16,5) and substituting $n = dl$ we obtain

$$\Phi_2(p) = \frac{\mathcal{R}_2^* D p}{\varphi(D)} \sum_{j=1,2} \mathcal{R}_{2j} \mathcal{S}_{2j} + o(p) \quad (16.12)$$

where

$$\mathcal{S}_{2j} = \sum_n \frac{b_1(n)}{n} \sum_{n=dl} \lambda_2(d) d^{\beta_j} \chi(l) \mathcal{M}_2(d, l; 1-\beta_j).$$

Let $\mathcal{M}_2^*(s)$ be given by

$$\begin{aligned} \mathcal{M}_2^*(s) &= \mathcal{M}_2(1, 1; s) \quad \text{if } \chi(2) \neq 1, \\ \mathcal{M}_2^*(s) &= 2 \prod_{q>2} \frac{1 - q^{-s-\beta_1}}{(1 - q^{-s})(1 - \chi(q)q^{-s})} \left(1 + \tilde{\lambda}_2(q, 1) \sum_r \frac{\xi_2(q^r; 1, 1)}{q^{rs}} \right) \quad \text{if } \chi(2) = 1. \end{aligned}$$

The following result will be proved in Appendix A.

Lemma 16.1. *Suppose $dl < PT^{-2}$ and $|s-1| < 5\alpha$. Then*

$$\mathcal{M}_2^*(s) = \mathfrak{p} + O(1/\mathcal{L}^8)$$

where

$$\mathfrak{p} = \prod_q \frac{1}{1 - \chi(q)q^{-1}} \left(1 - \frac{\chi(q)}{q-1} \right) \quad \text{if } \chi(2) \neq 1,$$

and

$$\mathfrak{p} = 2 \prod_{q>2} \frac{1}{1 - \chi(q)q^{-1}} \left(1 - \frac{\chi(q)}{q-1} \right) \quad \text{if } \chi(2) = 1.$$

As a direct consequence of Lemma 16.1 we have $|\mathcal{M}_2^*(s)| \gg 1$ if $|s - 1| < 5\alpha$. Thus we can write

$$\mathcal{S}_{2j} = \mathcal{M}_2^*(1 - \beta_j) \sum_n \frac{b_1(n)\varpi_{2j}(n)}{n} \quad (16.13)$$

with

$$\varpi_{2j}(n) = \sum_{n=dl} \lambda_2(d)d^{\beta_j} \chi(l) \frac{\mathcal{M}_2(d, l; 1 - \beta_j)}{\mathcal{M}_2^*(1 - \beta_j)}.$$

Note that $\varpi_{2j}(n)$ is multiplicative. In a way similar to the proof of (15.20) we have

$$\sum_n \frac{b_1(n)\varpi_{2j}(n)}{n} = \sum_{\substack{n_1 \in \mathcal{N}(\mathcal{Q}) \\ n_1 < T}} \frac{\varpi_{2j}(n_1)}{n_1} \sum_{(n, \mathcal{Q})=1} \frac{b_1(n_1 n)\varpi_{2j}(n)}{n} + O(D^{-c}).$$

Assume $n_1 \in \mathcal{N}(\mathcal{Q})$ and $n_1 < T$. For $(n, \mathcal{Q}) = 1$ we have

$$b_1(n_1 n) = \sum_{n_1=l_1 m_1} \sum_{n=l_1 m} (l_1 l)^{-\beta_3} g^*\left(\frac{T^2}{l_1 l}\right) b(m_1 m).$$

Hence

$$\sum_{(n, \mathcal{Q})=1} \frac{b_1(n_1 n)\varpi_{2j}(n)}{n} = \sum_{n_1=l_1 m_1} \sum_{(l, \mathcal{Q})=1} \frac{\varpi_{2j}(l)}{l(l_1 l)^{\beta_3}} g^*\left(\frac{T^2}{l_1 l}\right) \sum_{(m, \mathcal{Q})=1} \frac{b(m_1 m)\varpi_{2j}(m)}{m} + O(D^{-c}),$$

since the terms with $(l, m) > 1$ above contribute $\ll D^{-c}$. Further, for $(m, \mathcal{Q}) = 1$ and $m < P$ we have trivially

$$\varpi_{2j}(m) = \chi(m)\varrho_j^*(m) + O(\tau_2(m)D^{-c}).$$

Hence, for $m_1 < T$,

$$\sum_{(m, \mathcal{Q})=1} \frac{b(m_1 m)\varpi_{2j}(m)}{m} = \mathbf{e}_j \chi(m_1)\tau_2(m_1) + O(D^{-c})$$

by Lemma 15.1. It follows that

$$\sum_{(n, \mathcal{Q})=1} \frac{b_1(n_1 n)\varpi_{2j}(n)}{n} = \mathbf{e}_j \sum_{n_1=l_1 m_1} \chi(m_1)\tau_2(m_1) \sum_{(l, \mathcal{Q})=1} \frac{\varpi_{2j}(l)}{l(l_1 l)^{\beta_3}} g^*\left(\frac{T^2}{l_1 l}\right) + O(D^{-c}). \quad (16.14)$$

On the other hand, if $1 < l < T^5$ and $(l, \mathcal{Q}) = 1$, then for any $q|l$,

$$\varrho_{2j}(l) \ll \alpha_1 + O(\nu(q)).$$

Thus the innermost sum in (16.14) is equal to $1 + O(1/\mathcal{L}^7)$. It follows that

$$\sum_{(n, \mathcal{Q})=1} \frac{b_1(n_1 n) \varpi_{2j}(n)}{n} = \mathfrak{e}_j \sum_{m_1 | n_1} \chi(m_1) \tau_2(m_1) + O(1/\mathcal{L}^4).$$

Since $1 * \chi \tau_2 = \nu * \chi$ (here 1 denotes the arithmetic function identically equal to 1), it follows that

$$\sum_n \frac{b_1(n) \varpi_{2j}(n)}{n} = \mathfrak{e}_j \sum_{\substack{n_1 \in \mathcal{N}(\mathcal{Q}) \\ n_1 < T}} \frac{\varpi_{2j}(n_1) (\nu * \chi)(n_1)}{n_1} + O(1/\mathcal{L}). \quad (16.15)$$

On the right side above, the constraint $n_1 \in \mathcal{N}(\mathcal{Q})$ can be removed with an acceptable error.

The following lemma will be proved in Appendix A.

Lemma 16.2. *The function*

$$\mathcal{U}_{2j}(s) = \frac{1}{\zeta(s)^3 L(s, \chi)^3} \sum_n \frac{\varpi_{2j}(n) (\nu * \chi)(n)}{n^s}$$

is analytic and bounded for $\sigma > 9/10$. Further we have

$$\mathcal{U}_{2j}(1) = \frac{6}{\pi^2} \frac{\varphi(D)}{D \mathfrak{p}} \prod_{q|D} \frac{q}{q+1} + O(1/\mathcal{L}^4).$$

We have

$$\begin{aligned} \sum_{n < T} \frac{\varpi_{2j}(n) (\nu * \chi)(n)}{n} &= \sum_n \frac{\varpi_{2j}(n) (\nu * \chi)(n)}{n} g\left(\frac{T}{n}\right) + O(1/\mathcal{L}^{10}) \\ &= \frac{1}{2\pi i} \int_{(1)} \zeta(1+s)^3 L(1+s, \chi)^3 \mathcal{U}_{2j}(1+s) T^s \omega_1(s) \frac{ds}{s} + O(1/\mathcal{L}^{10}). \end{aligned}$$

The contour of integration is moved in the same way as in the proof of Lemma 8.4. Thus the right side above is, by Lemma 16.2, equal to

$$L'(1, \chi)^3 \mathcal{U}_{2j}(1) + O(1/\mathcal{L}^4) = \frac{\mathfrak{a}}{\mathfrak{p}} \frac{\varphi(D)}{D} L'(1, \chi) + O(1/\mathcal{L}^4).$$

Hence, by (16.15),

$$\sum_n \frac{b_1(n) \varpi_{2j}(n)}{n} = \frac{\mathfrak{a} \mathfrak{e}_j}{\mathfrak{p}} \frac{\varphi(D)}{D} L'(1, \chi) + O(1/\mathcal{L}^4).$$

Inserting this into (16.13) and applying Lemma 16.1 we obtain

$$\mathcal{S}_{2j} = \mathfrak{a}\mathfrak{e}_j \frac{\varphi(D)}{D} L'(1, \chi) + O(1/\mathcal{L}^4). \quad (16.16)$$

On the other hand, by Lemma 5.8 and direct calculation,

$$\mathcal{R}_2^* = \beta_1 + O(1/\mathcal{L}^{10}), \quad \mathcal{R}_{2j} = -\frac{1}{\beta_1 L'(1, \chi)} + O(\mathcal{L}^6), \quad j = 1, 2,$$

so that

$$\mathcal{R}_2^* \mathcal{R}_{2j} = -\frac{1}{L'(1, \chi)} + O(\mathcal{L}^{-1} L'(1, \chi)^{-1}).$$

This together with (16.16) and (16.12) yields

$$\Phi_2(p) = -(\mathfrak{e}_1 + \mathfrak{e}_2) \mathfrak{a}p + o(p).$$

Since $(pt_0)^{\beta_1} = -1 + O(\alpha_1)$, it follows by (16.2) that

$$\Phi_2 = (\mathfrak{e}_1 + \mathfrak{e}_2) \mathfrak{a}\mathcal{P} + o(\mathcal{P}). \quad (16.17)$$

17. Evaluation of Φ_3

Recall that Φ_3 is given by (13.10). Write

$$\mathcal{K}_3(s, \psi) = \frac{L(s + \beta_1, \psi)}{L(s, \psi)} B(s, \psi) G(s, \psi) N(s + \beta_2, \psi) N(s + \beta_3, \psi) F(1 - s, \bar{\psi})$$

and

$$I_4^\pm(\psi) = \frac{1}{2\pi i} \int_{\mathcal{J}(\pm\alpha)} \mathcal{K}_3(s, \psi) \omega(s) ds.$$

Similar to the treatment of Φ_2 , we have

$$\Phi_3 = \sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} (I_4^+(\psi) - I_4^-(\psi)) + O(\varepsilon). \quad (17.1)$$

To treat the sum of $I_4^+(\psi)$ we move the segment $\mathcal{J}(\alpha)$ to $\mathcal{J}(1)$, and then extend the sum over Ψ_1 to the sum over Ψ with an acceptable error. Hence

$$\sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} I_4^+(\psi) = \sum_{p \sim P} (pt_0)^{\beta_3} \Phi_3^+(p) + o(\mathcal{P}) \quad (17.2)$$

where

$$\Phi_3^+(p) = \sum_{\psi \bmod p}^* \frac{1}{2\pi i} \int_{\mathcal{J}(1)} \mathcal{K}_3(s, \psi) \omega(s) ds.$$

For $\sigma > 1$ we can write

$$\frac{L(s + \beta_1, \psi)}{L(s, \psi)} B(s, \psi) G(s, \psi) N(s + \beta_2, \psi) N(s + \beta_3, \psi) = \sum_m \frac{\nu^*(m) \psi(m)}{m^s}.$$

Thus, replacing the segment $\mathcal{J}(1)$ by the line $\sigma = 3/2$ and integrating term by term give

$$\Phi_3^+(p) = \sum_m \sum_{n < D^4} \frac{\nu^*(m) \nu(n)}{n} \left(\frac{n}{m}\right)^{s_0} \left(\sum_{\psi \bmod p}^* \psi(m) \bar{\psi}(n) \right) \exp \left\{ -\mathcal{L}_2^2 \log^2 \frac{n}{m} \right\} + O(\varepsilon).$$

By trivial estimation, the contribution from the terms with $m \neq n$ above is $o(p)$. Hence

$$\Phi_3^+(p) = p \sum_{n < D^4} \frac{\nu^*(n) \nu(n)}{n} + o(p). \quad (17.3)$$

Note that for $n < D^4$

$$\nu^*(n) = \sum_{n=n_1 \dots n_6} \kappa_2(n_1) \chi(n_2 n_3) (\varkappa_1(n_2) + \iota_2 \varkappa_2(n_2)) (\bar{\iota}_3 \varkappa_3(n_3) + \bar{\iota}_4 \varkappa_4(n_3)) \nu(n_4) \tilde{g}_2(n_5) \tilde{g}_3(n_6).$$

A detailed analysis shows that (17.2) remains valid if, in the expression for $\nu^*(n)$, the factor $\kappa_2(n_1)$ is replaced by $(1 * \mu)(n_1)$, the factor

$$\chi(n_2 n_3) (\varkappa_1(n_2) + \iota_2 \varkappa_2(n_2)) (\bar{\iota}_3 \varkappa_3(n_3) + \bar{\iota}_4 \varkappa_4(n_3))$$

is replaced by $\mathfrak{e}_0 \chi(n_2 n_3)$ with

$$\mathfrak{e}_0 = (\varkappa_1(1) + \iota_2 \varkappa_2(1)) (\bar{\iota}_3 \varkappa_3(1) + \bar{\iota}_4 \varkappa_4(1)), \quad (17.4)$$

and the factors $\tilde{g}_2(n_5)$ and $\tilde{g}_3(n_6)$ are replaced by 1 respectively. Since

$$1 * \mu * \chi * \chi * \nu * 1 * 1 = \nu,$$

it follows that

$$\Phi_3^+(p) = \mathfrak{e}_0 p \sum_{n < D^4} \frac{\nu(n)^2}{n} + o(p). \quad (17.5)$$

The following lemma will be proved in Appendix B.

Lemma 17.1. *We have*

$$\sum_{n < D^4} \frac{\nu(n)^2}{n} = \mathfrak{a} + o(1)$$

Since $(pt_0)^{\beta_3} = -1 + O(\alpha_1)$, it follows by (17.5), (17.2) and Lemma 17.1 that

$$\sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} I_3^+(\psi) = -\mathfrak{e}_0 \mathfrak{a} \mathcal{P} + o(\mathcal{P}). \quad (17.6)$$

To treat the sum involving $I_4^-(\psi)$ on $\mathcal{J}(-\alpha)$ we first apply Lemma 5.1 to obtain

$$\frac{L(s + \beta_1, \psi)}{L(s, \psi)} = (pt_0)^{-\beta_1} \frac{L(1 - s - \beta_1, \bar{\psi})}{L(1 - s, \bar{\psi})} (1 + O(\alpha^6))$$

for $\psi \in \Psi_1$ and $s \in \mathcal{J}(-\alpha)$. Thus, in a way similar to the proof of (15.4),

$$\sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} I_3^-(\psi) = \sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_2} \cdot \frac{1}{2\pi i} \int_{\mathcal{J}(-\alpha)} \mathcal{K}_3^*(s, \psi) \omega(s) ds + o(\mathcal{P})$$

where

$$\mathcal{K}_3^*(\psi) = \frac{L(1 - s - \beta_1, \bar{\psi})}{L(1 - s, \bar{\psi})} F(1 - s, \bar{\psi}) B(s, \psi) G(s, \psi) N(s + \beta_2, \psi) N(s + \beta_3, \psi).$$

Moving the segment $\mathcal{J}(-\alpha)$ to $\mathcal{J}(-1)$ and then extend the sum over Ψ_1 to then sum over Ψ we obtain

$$\sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} I_3^-(\psi) = \sum_{p \sim P} (pt_0)^{\beta_2} \Phi_3^-(p) + o(\mathcal{P}) \quad (17.7)$$

where

$$\Phi_3^-(p) = \sum_{\psi \bmod p}^* \frac{1}{2\pi i} \int_{\mathcal{J}(-1)} \mathcal{K}_3^*(s, \psi) \omega(s) ds.$$

Write

$$G(s, \psi) N(s + \beta_2, \psi) N(s + \beta_3, \psi) = \sum_l \frac{\nu_1^*(l) \psi(l)}{l^s},$$

so that

$$B(s, \psi) G(s, \psi) N(s + \beta_2, \psi) N(s + \beta_3, \psi) = \sum_n \frac{(b * \nu_1^*)(n) \psi(n)}{n^s}.$$

On the other hand, for $\sigma < 0$,

$$\frac{L(1 - s - \beta_1, \bar{\psi})}{L(1 - s, \bar{\psi})} = \sum_m \frac{\bar{\kappa}_2(m) \bar{\psi}(m)}{m^{1-s}}$$

with $\bar{\kappa}_2(m) = \overline{\kappa_2(m)}$, so that

$$\frac{L(1 - s - \beta_1, \bar{\psi})}{L(1 - s, \bar{\psi})} F(1 - s, \bar{\psi}) = \sum_n \frac{\varrho^*(n) \bar{\psi}(n)}{n^{1-s}}$$

with

$$\varrho^*(n) = \sum_{\substack{n=lm \\ l < D^4}} \nu(l) \bar{\kappa}_2(m).$$

Note that the arithmetic function $(b * \nu_1^*)(n)$ is supported on $n < PT^{-2}$. In a way similar to the proof of (17.3) we deduce that

$$\Phi_3^-(p) = p \sum_n \frac{(b * \nu_1^*)(n) \varrho^*(n)}{n} + o(p). \quad (17.8)$$

We have

$$\sum_n \frac{(b * \nu_1^*)(n) \varrho^*(n)}{n} = \sum_{l < D^4} \frac{\nu(l)}{l} \sum_m \frac{(b * \nu_1^*)(lm) \bar{\kappa}_2(m)}{m}.$$

The inner sum is equal to

$$\sum_{l=l_1 l_2} \sum_{m_1} \sum_{(m_2, l_1)=1} \frac{b(l_1 m_1) \nu_1^*(l_2 m_2) \bar{\kappa}_2(m_1 m_2)}{m_1 m_2}.$$

Note that $\nu_1^*(n)$ is supported on $n < T^5$. On the right side above, we can drop the terms with $m_2 > 1$ or $(m_1, \mathcal{Q}) > 1$ with an acceptable error. Hence

$$\sum_n \frac{(b * \nu_1^*)(n) \varrho^*(n)}{n} = \sum_{l < D^4} \frac{\nu(l)}{l} \sum_{l=l_1 l_2} \sum_{(m_1, \mathcal{Q})=1} \frac{b(l_1 m_1) \nu_1^*(l_2) \bar{\kappa}_2(m_1)}{m_1} + o(1).$$

If m_1 is square-free, then

$$\bar{\kappa}_2(m_1) = \mu(m_1) \varrho_1(m_1).$$

Hence, by Lemma 15.1 (see (15.)),

$$\sum_{(m_1, \mathcal{Q})=1} \frac{b(l_1 m_1) \bar{\kappa}_2(m_1)}{m_1} = \mathbf{e}_1 \chi(l_1) \tau_2(l_1) + O$$

It follows that

$$\sum_n \frac{(b * \nu_1^*)(n) \varrho^*(n)}{n} = \mathbf{e}_1 \sum_{l < D^4} \frac{\nu(l)}{l} \sum_{l=l_1 l_2} \chi(l_1) \tau_2(l_1) \nu_1^*(l_2) + o(1).$$

In a way similar to the proof of (17.5), by Lemma 17.1, we find that the right side is equal to

$$\mathbf{e}_1 \sum_{l < D^4} \frac{\nu(l)^2}{l} + o(1) = \mathbf{e}_1 \mathbf{a} + o(1).$$

Inserting this into (17.8) gives

$$\Phi_3^-(p) = \mathbf{e}_1 \mathbf{a} p + o(p).$$

Since $(pt_0)^{\beta_2} = 1 + O(\alpha_1)$, it follows by (17.7) that

$$\sum_{\psi \in \Psi_1} (p_\psi t_0)^{\beta_3} I_3^-(\psi) = \mathbf{e}_1 \mathbf{a} \mathcal{P} + o(\mathcal{P}). \quad (17.9)$$

Finally, from (17.1), (17.6) and (17.9) we conclude

$$\Phi_3 = -(\boldsymbol{\epsilon}_0 + \boldsymbol{\epsilon}_1)\boldsymbol{\alpha}\mathcal{P} + o(\mathcal{P}). \quad (17.10)$$

18. Proof of Proposition 2.5

By the discussion at the end of Section 2, it suffices to prove (2.32) and (2.33).

Proof of (2.32).

By (12.3), (12.17), (13.7), (15.24), (16.17) and (17.10),

$$\Xi_{13} = \mathbf{c}_3\boldsymbol{\alpha}\mathcal{P} \quad (18.1)$$

with

$$\mathbf{c}_3 = -i(3\boldsymbol{\epsilon}_1 + 3\boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_3 + \boldsymbol{\epsilon}_0) + e_1^* + 2e_2^* + \epsilon.$$

In view of (15.), we can write

$$\boldsymbol{\epsilon}_j = \boldsymbol{\epsilon}'_j - \boldsymbol{\epsilon}''_j, \quad 1 \leq j \leq 3$$

with

$$\begin{aligned} \boldsymbol{\epsilon}'_j &= (e'_{1j} + \iota_2 e_{2j})(\bar{\iota}_3 e_{3j} + \bar{\iota}_4 e_{2j}) \\ \boldsymbol{\epsilon}''_j &= e''_{1j}(\bar{\iota}_3 e_{3j} + \bar{\iota}_4 e_{2j}). \end{aligned}$$

By calculation (there is a theoretical interpretation),

$$i(3\boldsymbol{\epsilon}''_1 + 3\boldsymbol{\epsilon}''_2 + \boldsymbol{\epsilon}''_3) + e_1^* = \epsilon.$$

Hence

$$\mathbf{c}_3 = -i(3\boldsymbol{\epsilon}'_1 + 3\boldsymbol{\epsilon}'_2 + \boldsymbol{\epsilon}'_3 + \boldsymbol{\epsilon}_0) + 2e_2^* + 2\epsilon.$$

Direct calculation shows that

$$\Re\{\mathbf{c}_3\} < -6.9951 \quad (18.2)$$

It follows from (8.24), (9.8) and (18.2) that

$$\mathbf{c}_1 + \mathbf{c}_2 + 2\Re\{\mathbf{c}_3\} < 0.001.$$

This with together (8.23), (9.7) and (18.1) yields (2.32).

Proof of (2.33).

By Lemma 8.1,

$$\sum_{\psi \in \Psi_1} \sum_{\rho \in \mathcal{Z}(\psi)} |J_1(\rho, \psi)|^2 \omega(\rho) = 2\Re\{\Theta_1(\mathbf{a}_{23}, \mathbf{a}_{23})\} + o(P). \quad (18.3)$$

We have

$$\begin{aligned} S_j(\mathbf{a}_{23}, \mathbf{a}_{23}) &= \sum_r \sum_d \frac{|\chi(d)| |\mu\chi(r)| \lambda_{0j}(dr)}{dr\varphi(r)} \left(\sum_m \frac{\chi(m) \tilde{f}(\log(drm)/\log P)}{m^{1-\beta_j}} \right) \\ &\quad \times \left(\sum_n \frac{\chi(n) \tilde{f}(\log(drn)/\log P) \xi_{0j}(n; d, r)}{n} \right). \end{aligned}$$

The right side is split into three sums according to

$$dr < P^{0.5}, \quad P^{0.5} \leq dr < P^{0.502}, \quad P^{0.502} \leq dr < P^{0.504}.$$

By the discussion in Section 8 and 10, the sum over $dr < P^{0.5}$ contributes $o(\alpha)$; the sum over $P^{0.5} \leq dr < P^{0.502}$ is equal to

$$\left(\frac{500L'(1, \chi)}{\log P} \right)^2 \sum_{P^{0.5} \leq n < P^{0.502}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} (-1 - \beta_j \log(n/P^{0.5})) (-1 + \mathcal{Y}_{1j}(n)) + o(\alpha);$$

the sum over $P^{0.502} \leq dr < P^{0.504}$ is equal to

$$\left(\frac{500L'(1, \chi)}{\log P} \right)^2 \sum_{P^{0.502} \leq n < P^{0.504}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} (1 - \beta_j \log(P^{0.504}/n)) (1 + \mathcal{Y}_{1j}(n)) + o(\alpha).$$

The factors $\beta_j \log(n/P^{0.5})$, $\beta_j \log(P^{0.504}/n)$ and $\mathcal{Y}_{1j}(n)$ in the above sums make minor contribution. If we disregard these factors, the major contribution to $S_j(\mathbf{a}_{23}, \mathbf{a}_{23})$ will be

$$\left(\frac{500L'(1, \chi)}{\log P} \right)^2 \sum_{P^{0.5} \leq n < P^{0.504}} \frac{|\chi(n)| \lambda_{0j}(n)}{\varphi(n)} = \frac{1000\mathbf{a}}{\log P} + o(\alpha).$$

Thus we have the crude bound

$$\Re\{S_j(\mathbf{a}_{23}, \mathbf{a}_{23})\} < \frac{1100\mathbf{a}}{\log P},$$

so that

$$\Re\left\{ \frac{1}{2\alpha} S_1(\mathbf{a}_{23}, \mathbf{a}_{23}) + \frac{2}{\alpha} S_2(\mathbf{a}_{23}, \mathbf{a}_{23}) + \frac{3}{2\alpha} S_3(\mathbf{a}_{23}, \mathbf{a}_{23}) \right\} < \frac{4400\mathbf{a}}{\pi}.$$

This yields (2.33) by Lemma 8.1 and Proposition 7.1. \square

Appendix A. Some Euler Products

This appendix is devoted to proving Lemma 8.3, 15.2, 15.3, 16.1 and 16.2. For notational simplicity we shall write

$$u = q^{-1}, \quad v = \chi(q).$$

Proof of Lemma 8.3. Note that

$$\lambda(km, s) = \lambda(k, s)\tilde{\lambda}(m, k; s)$$

with

$$\tilde{\lambda}(m, k; s) = \prod_{\substack{q|m \\ (q,k)=1}} \frac{(1 - q^{-s-\beta_1})(1 - q^{-s-\beta_2})(1 - q^{-s-\beta_3})}{1 - q^{-s}}.$$

For given d and h , the function $\tilde{\lambda}(m, h, d; 1 - \beta_j)\xi_j(m; d, h)$ is multiplicative in m , and $\tilde{\lambda}(q^r, k; s) = \tilde{\lambda}(q, k; s)$ for any r . Thus, for $\sigma > 1$,

$$\mathcal{U}_j(d, h; s) = \lambda(dh, 1 - \beta_j) \prod_q \mathcal{T}_j(d, 1, s; q)$$

with

$$\mathcal{T}_j(d, h, s; q) = \frac{(1 - \chi(q)q^{-s-\beta_{j+1}})(1 - \chi(q)q^{-s-\beta_{j+2}})}{1 - \chi(q)q^{-s}} \left(1 + \tilde{\lambda}(q, dh; 1 - \beta_j) \sum_r \frac{\chi(q^r)\xi_j(q^r; d, h)}{q^{rs}} \right).$$

Since

$$\begin{aligned} \tilde{\lambda}(q; d, 1 - \beta_j)\chi(q)\xi_j(q^r; d, h) &= \chi(q)(\kappa(q) - q^{-\beta_j}) + O(q^{-1}) \\ &= \chi(q)(q^{-\beta_{j+1}} + q^{-\beta_{j+2}} - 1) + O(q^{-1}), \end{aligned}$$

we have $\mathcal{T}_j(d, h, s; q) = 1 + O(q^{-9/5})$ for $\sigma > 9/10$, so $\mathcal{U}_{j\mu}(d, s)$ is analytic in this region, and

$$\mathcal{U}_j(d, h; s) = \lambda(dh, 1 - \beta_j) \prod_{q < D} \mathcal{T}_j(d, h_1, s; q) + O(D^{-c}).$$

Since

$$\lambda(dh, 1 - \beta_j) = \frac{\varphi(dh)^2}{(dh_1)^2} + O(\alpha_1)$$

and $\mathcal{T}_j(d, h, s; q) = 1$ if $q|D$, the proof is reduced to showing that

$$\mathcal{T}_j(d, h, s; q) = 1 + O(\alpha \log q/q) \quad \text{if } (q, dh_1) = 1, \tag{A.1}$$

$$\mathcal{T}_j(d, h, s; q) = \frac{1}{1 - vu} + O(\alpha \log q/q) \quad \text{if } q|h_1, \quad (\text{A.2})$$

and

$$\mathcal{T}_j(d, h, s; q) = \frac{1 - u - vu}{(1 - vu)(1 - u)} + O(\alpha \log q/q) \quad \text{if } q|d \quad (q, h_1) = 1, \quad (\text{A.3})$$

provided

$$|s - 1| < 5\alpha, \quad q < D \quad \text{and} \quad (q, D) = 1,$$

which are henceforth assumed. We discuss in three cases.

Case 1. $(q, dh) = 1$.

We have

$$\xi_j(q^r, dh) = \tilde{\kappa}(q^r; dh, 1 - \beta_j) - \frac{\kappa(q^{r-1})q^{1-\beta_j}}{q - 1}.$$

because $\tilde{\kappa}(q^{r-1}; dq, 1 - \beta_j) = \kappa(q^{r-1})$. Note that

$$\kappa(q^r) = \sum_{\mu_1 + \mu_2 + \mu_3 = r} q^{\mu_1\beta_1 + \mu_2\beta_2 + \mu_3\beta_3} - \sum_{\mu_1 + \mu_2 + \mu_3 = r-1} q^{\mu_1\beta_1 + \mu_2\beta_2 + \mu_3\beta_3} = \tau_2(q^r)(1 + O(\alpha r \log q))$$

where μ_1, μ_2 and μ_3 run through non-negative integers. Hence

$$\xi_j(q^r, dh) = \sum_{\eta=0}^{\infty} (r + \eta + 1)u^\eta - \frac{r}{1 - u} + O(\alpha r \log q/q) = \frac{1}{(1 - u)^2} + O(\alpha r \log q/q).$$

It follows that

$$\sum_r \frac{\chi(q^r)\xi_j(q^r; d, h)}{q^{rs}} = \frac{1}{(1 - u)^2} \left(\frac{1}{1 - vu} - 1 \right) + O(\alpha \log q/q).$$

This together with the relations

$$\tilde{\lambda}(q, dh; 1 - \beta_j) = (1 - u)^2 + O(\alpha \log q/q)$$

and

$$\frac{(1 - \chi(q)q^{-s-\beta_{j+1}})(1 - \chi(q)q^{-s-\beta_{j+2}})}{1 - \chi(q)q^{-s}} = 1 - vu + O(\alpha \log q/q),$$

yields (A.1).

Case 2. $q|h$.

We have

$$\xi_j(q^r, dh) = \kappa(q^r) = r + 1 + O(\alpha r \log q/q),$$

so that

$$\sum_r \frac{\chi(q^r) \xi_j(q^r; d, h)}{q^{rs}} = \frac{1}{(1-vu)^2} - 1 + O(\alpha \log q/q).$$

This yields (A.2) since $\tilde{\lambda}(q, dh; 1 - \beta_j) = 1$.

Case 3. $q|d$, $(q, h_1) = 1$.

We also have $\tilde{\lambda}(q, dh; 1 - \beta_j) = 1$ and

$$\xi_j(q^r, dh) = \kappa(q^r) - \frac{\kappa(q^{r-1})q}{q-1} = r + 1 - \frac{r}{1-u} + O(\alpha r \log q/q),$$

so that

$$\sum_r \frac{\chi(q^r) \xi_j(q^r; d, h)}{q^{rs}} = \frac{1}{(1-vu)^2} - 1 - \frac{vu}{(1-u)(1-vu)^2} + O(\alpha \log q/q).$$

This yields (A.3). \square

To prove Lemma 15.2 and 15.3 we need several prerequisite results. In what follows assume $dl < P_2^2$, $(dl, D) = 1$ and $|s-1| < 5\alpha$. We have

$$\begin{aligned} \mathcal{M}_1(d, l; s) &= \prod_{q < D} \frac{(1 - q^{-s-\beta_1})(1 - q^{-s-\beta_2})}{(1 - q^{-s})(1 - \chi(q)q^{-s})} \left(1 + \tilde{\lambda}_1(q, d) \sum_r \frac{\xi_1(q^r; d, l)}{q^{rs}} \right) \\ &\quad \times (1 + O(D^{-c})). \end{aligned} \quad (\text{A.4})$$

Note that

$$\tilde{\kappa}_1(q^{r-1}, dq) = \kappa_1(q^{r-1}),$$

so that

$$\xi_1(q^r; d, l) = \begin{cases} \tilde{\kappa}_1(q^r, d) - \frac{\chi(q)q}{q-1} \kappa_1(q^{r-1}) & \text{if } (q, l) = 1 \\ \tilde{\kappa}_1(q^r, d) & \text{if } q|l \end{cases} \quad (\text{A.5})$$

for any q, d, l and r .

Proof of Lemma 15.2. By (A.4), (A.5) and the relations

$$\tilde{\lambda}_1(q^r, 1) = \lambda_1(q), \quad \frac{(1 - q^{-s-\beta_1})(1 - q^{-s-\beta_2})}{(1 - q^{-s})(1 - \chi(q)q^{-s})} = \frac{1 - q^{-1}}{1 - \chi(q)q^{-1}} + O\left(\frac{\alpha \log q}{q}\right),$$

it suffices to show that

$$\begin{aligned} 1 + \lambda_1(q) \sum_r \frac{1}{q^{rs}} \left(\tilde{\kappa}_1(q^r, 1) - \frac{\chi(q)q}{q-1} \kappa_1(q^{r-1}) \right) &= \frac{1 - \chi(q)q^{-1}}{1 - q^{-1}} \frac{1 - \chi(q)q^{-2}}{1 - q^{-2}} \\ &\quad + O\left(\frac{\alpha \log q}{q}\right) \end{aligned} \quad (\text{A.6})$$

if $q < D$, $(q, D) = 1$, and

$$1 + \lambda_1(q) \sum_r \frac{1}{q^{rs}} \left(\tilde{\kappa}_1(q^r, 1) - \frac{\chi(q)q}{q-1} \kappa_1(q^{r-1}) \right) = \frac{1}{1-q^{-1}} + O\left(\frac{\alpha \log q}{q}\right) \quad (\text{A.7})$$

if $q|D$.

Assume $q < D$. Since

$$\kappa_1(q^r) = \sum_{\mu_1+\mu_2=r} q^{-\mu_1\beta_1-\mu_2\beta_2} - \sum_{\mu_1+\mu_2=r-1} q^{-\mu_1\beta_1-\mu_2\beta_2},$$

where μ_1 and μ_2 run through non-negative integers, it follows that

$$\kappa_1(q^r) = 1 + O(\alpha r \log q)$$

if $q^r < P$. Hence

$$\tilde{\kappa}_1(q^r, 1) = \sum_{h \in \mathcal{N}(q)} \frac{\kappa_1(q^r h) \chi(h)}{h} = \sum_{h \in \mathcal{N}(q)} \frac{\chi(h)}{h} + O(\alpha r \log q) = \frac{1}{1-vu} + O(\alpha r \log q).$$

It follows that

$$\sum_r \frac{1}{q^{rs}} \left(\tilde{\kappa}_1(q^r, 1) - \frac{\chi(q)q}{q-1} \kappa_1(q^{r-1}) \right) = \frac{u}{1-u} \left(\frac{1}{1-vu} - \frac{v}{1-u} \right) + O(\alpha \log q/q).$$

This together with the relation

$$\lambda_1(q) = 1 - vu + O(\alpha \log q/q)$$

yields

$$1 + \lambda_1(q) \sum_r \frac{1}{q^{rs}} \left(\tilde{\kappa}_1(q^r, 1) - \frac{\chi(q)q}{q-1} \kappa_1(q^{r-1}) \right) = \frac{1}{1-u} - \frac{uv(1-vu)}{(1-u)^2} + O(\alpha \log q/q).$$

It is direct to verify that

$$\frac{1}{1-u} - \frac{uv(1-vu)}{(1-u)^2} = \frac{1 - \chi(q)q^{-1}}{1-q^{-1}} \frac{1 - \chi(q)q^{-2}}{1-q^{-2}}$$

if $\chi(q) = \pm 1$, and

$$\frac{1}{1-u} - \frac{uv(1-vu)}{(1-u)^2} = \frac{1}{1-q^{-1}}$$

if $\chi(q) = 0$, whence (A.6) and (A.7) follow. \square

Proof of Lemma 15.3. We give a sketch only, as the situation is similar to Lemma 15.2. it can be verified that the factor $\mathcal{U}_{1j}(q, s)$ in the Euler product representation

$$\mathcal{U}_{1j}(s) = \prod_q \mathcal{U}_{1j}(q, s)$$

satisfies $\mathcal{U}_{1j}(q, s) = 1 + O(q^{-2\sigma})$ if $(q, D) = 1$. Further, in the case $q \leq D$ we have

$$\mathcal{U}_{1j}(q, 1) = (1 - q^{-1})^2 + O(\alpha_1/q) \quad \text{if } q|D,$$

and

$$\mathcal{U}_{1j}(q, 1) = \frac{(1 - q^{-2})^2}{1 - \chi(q)q^{-2}} + O(\alpha_1/q) \quad \text{if } (q, D) = 1.$$

This completes the proof. \square

Proof of Lemma 16.1. For any q, r, d and l we have

$$\xi_2(q^r; d, l) = \begin{cases} \tilde{\kappa}_2(q^r; d) - \frac{\chi(q)q}{q-1}\kappa_2(q^{r-1}) & \text{if } (q, l) = 1 \\ \tilde{\kappa}_2(q^r; d) & \text{if } q|l, \end{cases}$$

$$|\kappa_2(q^r)| = |q^{-\beta_1} - 1| \ll \alpha \log q, \quad \tilde{\lambda}_2(q, d) = 1 + O(\alpha \log q/q).$$

Hence

$$1 + \tilde{\lambda}_2(q, d) \sum_r \frac{\xi_2(q^r; d, l)}{q^{rs}} = 1 - \frac{\chi(q)}{q-1} + O(\alpha \log q/q) \quad \text{if } (q, l) = 1$$

and

$$1 + \tilde{\lambda}_2(q, d) \sum_r \frac{\xi_2(q^r; d, l)}{q^{rs}} = 1 + O(\alpha \log q/q) \quad \text{if } q|l.$$

On the other hand we have

$$\frac{1 - q^{-s-\beta_1}}{(1 - q^{-s})(1 - \chi(q)q^{-s})} = \frac{1}{1 - \chi(q)q^{-1}} + O(\alpha \log q/q).$$

It follows that

$$\mathcal{M}_2(d, l; s) = \prod_{\substack{q < D \\ (q, l) = 1}} \frac{1}{1 - \chi(q)q^{-1}} \left(1 - \frac{\chi(q)}{q-1}\right) \prod_{\substack{q < D \\ q|l}} \frac{1}{1 - \chi(q)q^{-1}} + O(\alpha_1).$$

It is direct to verify that in either case the assertion holds. \square

Proof of Lemma 16.2. We give a sketch only. If $dl < D$, $(dl, D) = 1$ and $|s - 1| \leq 5\alpha$, then

$$\frac{\mathcal{M}_2(d, l; s)}{\mathcal{M}_2^*(s)} = \Pi_2(l) + O(\alpha_1 \tau_2(dl))$$

with

$$\begin{aligned}\Pi_2(l) &= \prod_{q|l} \left(1 - \frac{\chi(q)}{q-1}\right)^{-1} \quad \text{if } \chi(2) \neq 1, \\ \Pi_2(l) &= \prod_{\substack{q|l \\ q>2}} \left(1 - \frac{\chi(q)}{q-1}\right)^{-1} \quad \text{if } \chi(2) = 1, 2|l, \\ \Pi_2(l) &= 0 \quad \text{if } \chi(2) = (l, 2) = 1.\end{aligned}$$

The assertion follows by discussing the cases $\chi(2) \neq 1$ and $\chi(2) = 1$ respectively. \square

Appendix B. Some arithmetic sums

Proof of Lemma 15.1. Put

$$\varrho_j(n) = \sum_{d|n} \mu(d) d^{\beta_j}.$$

First we claim that

$$\sum_{\substack{n < P \\ (n, \mathcal{Q})=1}} \frac{|\varrho_j(n) - \varrho_j^*(n)|}{n} \ll \mathcal{L}^{-8}. \quad (\text{B.1})$$

Since $\chi = \mu * \nu$, it follows that

$$\chi(d) = \mu(d) + O\left(\sum_{\substack{h|d \\ h>1}} \nu(h)\right).$$

Hence

$$\varrho_j(n) - \varrho_j^*(n) \ll \sum_{\substack{h|n \\ h>1}} \nu(h) \tau_2(n/h).$$

If $h > 1$ and $(h, \mathcal{Q}) = 1$, then $h > D^4$. Hence, by substituting $n = hm$,

$$\sum_{\substack{n < P \\ (n, \mathcal{Q})=1}} \frac{|\varrho_j(n) - \varrho_j^*(n)|}{n} \ll \sum_{h>D^4} \frac{\nu(h)}{h} \sum_{m < P/h} \frac{\tau_2(m)}{m} \ll (\log P)^2 \sum_{D^4 < h < P} \frac{\nu(h)}{h}.$$

This together with Lemma 3.2 yields (B.1).

Next we claim that

$$\sum_{\substack{n < P \\ (n, \mathcal{Q})>1}} \frac{|\varrho_j(n)|}{n} \ll \mathcal{L}^{-8}. \quad (\text{B.2})$$

Note that $\varrho_j(q^r) = \varrho_j(q)$ for any r , and $\varrho_j(q) \ll \alpha \log q$ if $q < P$. Thus the left side above is

$$\leq \left(\sum_{q < D^4} \frac{|\varrho_j(q)|}{q} \right) \prod_{q < P} (1 + |\varrho_j(q)|/q + O(1/q^2)) \ll \alpha \sum_{q < D^4} \frac{\log q}{q}.$$

This yields (B.2).

By (B.1) and (B.2), for $\mu = 2, 3$,

$$\sum_{(l, \mathcal{Q})=1} \frac{\varkappa_\mu(l_1 l) \varrho_j^*(l)}{l} = \sum_l \frac{\varkappa_\mu(l_1 l) \varrho_j(l)}{l} + O(\mathcal{L}^{-8}).$$

We proceed to prove the assertion with $\mu = 2$. Since

$$\sum_l \frac{\varrho_j(l)}{l^s} = \frac{\zeta(s)}{\zeta(s - \beta_j)}$$

for $\sigma > 1$ and

$$\varkappa_2(m) = \frac{1}{2\pi i} \int_{(1)} \frac{(P_2/m)^s}{(\log P_2)(s - \beta_7)^2} ds,$$

it follows that

$$\sum_l \frac{\varkappa_2(l_1 l) \varrho_j(l)}{l} = \frac{1}{2\pi i} \int_{(1)} \frac{\zeta(1+s)}{\zeta(1+s - \beta_j)} \frac{(P_2/l_1)^s}{(\log P_2)(s - \beta_7)^2} ds.$$

In a way similar to the proof of Lemma 8.1, we see that the right side is equal to the sum of the residues of the integrand at $s = 0$ and $s = \beta_7$ plus an acceptable error. By direct calculation, the residue at $s = 0$ is

$$-\frac{\beta_j}{\beta_7^2 \log P_2} = -\frac{8j}{25\pi i} + O(\alpha_1);$$

the residue at $s = \beta_7$ is, by the Cauchy integral formula,

$$\begin{aligned} & \frac{\zeta(1 + \beta_7)}{\zeta(1 + \beta_7 - \beta_j)} \frac{\log(P_2/l_1)}{\log P_2} \left(\frac{P_2}{l_1}\right)^{\beta_7} + \frac{1}{\log P_2} \left(\frac{P_2}{l_1}\right)^{\beta_7} \frac{d}{ds} \frac{\zeta(1+s)}{\zeta(1+s - \beta_j)} \Big|_{s=\beta_7} \\ & = \left(1 - \frac{2j}{5} + \frac{8j}{25\pi i}\right) \exp\{5\pi i/4\} + O(\alpha_1). \end{aligned}$$

These together complete the proof in case $\mu = 2$. In case $\mu = 3$ the proof can be obtained with β_6 and P_3 in place of β_7 and P_2 respectively. The same argument also gives

$$\sum_l \frac{\varkappa_1(l_1 l) \varrho_j(l)}{l} = \left(1 - \frac{2j}{3} + \frac{j}{1.134\pi i}\right) \exp\{0.756\pi i\} - \frac{j}{1.134\pi i} + O(\alpha_1).$$

For $\mu = 1$ the proof is therefore reduced to showing that

$$\sum_{l > P^{12}/l_1} \frac{\varkappa_1(l_1 l) \varrho_j(l)}{l} = \frac{j}{0.756} \int_0^{0.004} (\exp\{(3/2)(0.504 - z)\pi i\} - \exp\{(3/4)\pi i\}) dz + O(\alpha_1). \quad (B.3)$$

By (4.2) and (4.3), the left side of (B.3) is equal to

$$\sum_l \frac{\varrho_j(l)}{l} \left(\frac{P_1}{l_1 l}\right)^{\beta_6} \frac{1}{0.504} \int_{0.5}^{0.504} \left\{ g\left(\frac{Pz}{l_1 l}\right) - g\left(\frac{P^{0.5}}{l_1 l}\right) \right\} dz + O(\alpha_1).$$

By a change of variable, for $0.5 \leq z \leq 0.504$,

$$\left(\frac{P_1}{y}\right)^{\beta_6} \left\{ g\left(\frac{Pz}{y}\right) - g\left(\frac{P^{0.5}}{y}\right) \right\} = \frac{1}{2\pi i} \int_{(1)} (P^{\beta_6(0.504-z)} P^{zs} - P^{0.004\beta_6} P^{0.5s}) \frac{\omega_1(s - \beta_6) ds}{y^s (s - \beta_6)}.$$

Hence, in a way similar to the proof of, we find that the left side of (B.3) is

$$\begin{aligned} & \frac{1}{0.504} \int_{0.5}^{0.504} \left\{ \frac{1}{2\pi i} \int_{(1)} (P^{\beta_6(0.504-z)} P^{zs} - P^{0.004\beta_6} P^{0.5s}) \frac{\zeta(1+s)}{\zeta(1+s-\beta_j)} \frac{\omega_1(s - \beta_6) ds}{l_1^s (s - \beta_6)} \right\} dz + O \\ &= \frac{1}{0.504} \frac{\beta_j}{\beta_6} \int_{0.5}^{0.504} (P^{\beta_6(0.504-z)} - P^{0.004\beta_6}) dz + O(\alpha_1). \quad \square \end{aligned}$$

Proof of Lemma 17.1. By Lemma 3.1,

$$\sum_{n < D^4} \frac{\nu(n)^2}{n} = \sum_n \frac{\nu(n)^2}{n} g\left(\frac{T}{n}\right) + o(1).$$

The sum on the right side is equal to

$$\frac{1}{2\pi i} \int_{(1)} \left(\sum_n \frac{\nu(n)^2}{n^{1+s}} \right) \frac{T^s \omega_1(s)}{s} ds.$$

Assume $\sigma > 1$. We have

$$\sum_n \frac{\nu(n)^2}{n^s} = \prod_p \left(1 + \sum_r \frac{\nu(p^r)^2}{p^{rs}} \right).$$

If $\chi(p) = 1$, then (see [19, (1.2.10)])

$$1 + \sum_r \frac{\nu(p^r)^2}{p^{rs}} = (1 - p^{-s})^{-4} (1 - p^{-2s}) = (1 - p^{-s})^{-2} (1 - \chi(p)p^{-s})^{-2} (1 - p^{-2s});$$

if $\chi(p) = -1$, then

$$1 + \sum_r \frac{\nu(p^r)^2}{p^{rs}} = 1 + \sum_r \frac{1}{p^{2rs}} = (1 - p^{-2s})^{-1} = (1 - p^{-s})^{-2}(1 - \chi(p)p^{-s})^{-2}(1 - p^{-2s});$$

if $\chi(p) = 0$, then

$$1 + \sum_r \frac{\nu(p^r)^2}{p^{rs}} = 1 + \sum_r \frac{1}{p^{rs}} = (1 - p^{-s})^{-1} = (1 - p^{-s})^{-2}(1 - \chi(p)p^{-s})^{-2}(1 - p^{-s}).$$

Hence

$$\sum_n \frac{\nu(n)^2}{n^s} = \zeta(s)^2 L(s, \chi)^2 \prod_{(p,D)=1} (1 - p^{-2s}) \prod_{p|D} (1 - p^{-s}).$$

In a way similar to the proof of, by (A) and simple estimate, we find that the integral (14) is equal to the residue of the function

$$\zeta(1+s)^2 L(1+s, \chi)^2 \prod_{(p,D)=1} (1 - p^{-2(1+s)}) \prod_{p|D} (1 - p^{-(1+s)}) \frac{T^s \omega_1(s)}{s}$$

at $s = 0$, plus an acceptable error O , which is equal to

$$L'(1, \chi)^2 \prod_p (1 - p^{-2}) \prod_{p|D} (1 - p^{-1})(1 - p^{-2})^{-1} + o(1) = \mathfrak{a} + o(1). \quad \square$$

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